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Dynamics of Entry and Exit of Product Varieties

– what evolution dynamics can account for the empirical regularities?

Martin Andersson, Börje Johansson and Kristofer Månsson

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DYNAMICS OF ENTRY AND EXIT OF PRODUCT VARIETIES

- what evolution dynamics can account for the empirical regularities?

Martin Andersson[∅], Börje Johansson[∅] and Kristofer Månsson[⊕]

Abstract

Firm-level heterogeneity is substantial even in narrowly defined industries. This paper focuses on formulating evolution dynamics which can account for the observed heterogeneity and its maintenance. Based on examination of data on Swedish firm' supply pattern to different markets over time, we present a parsimonious model that has the ambition to capture the picture of heterogeneous firms, while accommodating the simultaneous exit and entry of destination varieties in firms' supply pattern. The model assumes both scale economies of firms and path-dependence, where the latter is manifested in such a way that the arrival rate of innovation ideas to an individual firm is a function of each firm's stock of varieties at every given point in time. The path-dependence phenomenon is an "explosive" non-linearity, whereas conservation mechanisms include development of demand and exit of established varieties. The described path dependence explains the skewed distribution of varieties across firms, but the question of what keeps the "equilibrium" away from competitive exclusion where only few large firms remain. We make use of simulations to depict and assess the innovation dynamics of the proposed model.

Keywords: innovation, path-dependence, firm-level heterogeneity, evolution dynamics
JEL: C16, F14, L25, O33

[∅]Jönköping International Business School, Jönköping, and Centre of Excellence for Science and Innovation Studies (CESIS), Royal Institute of Technology, Stockholm. E-mail: martin.andersson@ihh.hj.se, borje.johansson@ihh.hj.se
[⊕] Jönköping International Business School, Jönköping E-mail: kristofer.mansson@ihh.hj.se

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1. INTRODUCTION

Standard models in economics tend to focus on the decisions and behaviors of a representative firm. The underlying conjecture is that firms operating in a similar environment and industry, i.e. facing similar external conditions, are 'forced' to behave in an alike manner. This is assumed to reduce heterogeneity amongst firms, such that firms within the same sector develop similar characteristics. However, even within narrowly defined industries, the level of inter-firm heterogeneity in terms of performance and production attributes is substantial. In summarizing their experience with working with firm-level data, Griliches and Mairesse (1995, p.23) beautifully describe it as follows:

> "We started our work in this area with the hope that micro-data may be the answer to the various difficulties encountered at the aggregate level, primarily because this is the level which our theories claim to comprehend, and because we believed that this will reduce multicollinearity and provide us with more identifying variance. We also thought that one could reduce aggregation biases by reducing the heterogeneity as one goes from such general mixtures as "total manufacturing" to something more coherent, such as "petroleum refining" or the "manufacture of cement". But something like Mandelbrot's fractals phenomenon seems to be at work here also: the observed variability-heterogeneity does not really decline as we cut our data finer and finer. There is a sense in which different bakeries are just as much different from each other, as the steel industry is from the machinery industry."

The described experience certainly applies to other scholars working with firm-level data from different countries and time periods. Heterogeneity among firms is the norm rather than the exception, even in well-defined sectors. Longitudinal data also show that heterogeneity is persistent over time, with some firms having a persistent productivity and technological advantage over others (cf. Cantwell and Andersen 1996).

The level of heterogeneity revealed by firm-level data gives of course rise to several questions.[*] In this paper, we focus on the question of what kind of evolution dynamics at the micro-level that can account for the observed heterogeneity and its maintenance. Our contribution is divided into three parts. First, using observations from a large Swedish database describing over a sequence of years individual firms' sales of different product varieties on different markets, we illustrate firm-level heterogeneity and its persistence over time. The level of detail of the data allow us to enrich the analysis of firm-level heterogeneity beyond the typical focus on skewed distributions of firm size (e.g. Ijiri and Simon 1977, Axtell 2001). We depict heterogeneity across firms in terms of observable features of firms' actual sales of different products on different markets in each time period, and also assess dynamics of firms' supply pattern over time. We demonstrate (i) the coexistence of large multi-variety and multi-market firms, and single-variety and single-market firms in the same product

[*] For instance, Griliches and Mairesse (1995, p.23) remark; "we need a richer theoretical framework to help us understand why firms are different, not only in their capital-labor ratios, but also in the product mix that they produce, the quality of their workforce, the technologies they use, their organizational structures, the markets that they serve."

group, and (ii) the intense exit and entry of varieties that firms' supply in each product group. The analysis shows that there is significant gross dynamics of entry and exit of products across firms, despite a persistent skewed distribution of firms with different products and variety stocks, i.e. the invariant skewed distribution of variety stocks across firms conceals significant dynamics in firms' supply pattern. Second, guided by the empirical regularities, we present a parsimonious model that has the ambition to capture the picture of heterogeneous firms as well as accommodate the simultaneous exit and entry of destination varieties in firms' product lines. In the model, innovation ideas arrive to existing as well as potential entrepreneurs, and the ideas are materialized in the form of new destination varieties. The model assumes both scale economies and path-dependence, such that the arrival rate of innovation ideas to an individual firm is a function of each firm's stock of varieties at every given point in time. The arrival rate is thus determined by state-dependent Poisson processed that are individual to each firm. The path-dependence phenomenon is an "explosive" non-linearity, whereas conservation mechanisms include development of demand and exit of established varieties. Third, we make use of simulations to assess (i) whether the described micro-dynamics imply that initially skewed distributions, i.e. micro heterogeneity, remain, (ii) whether the modeling assumptions imply that an initially homogeneous distribution become skewed over time and (iii) if there are 'critical values' at which the system explodes through bifurcation.

The remainder of the paper is organized in the following fashion: Section 2 presents the data and demonstrate the level of heterogeneity amongst firms. Section 3 outlines the model, and Section 4 presents simulation based on the model. Section 5 concludes.

2. THE EXTENT OF FIRM-LEVEL HETEROGENEITY – evidence from Swedish data

The purpose of this section is to illustrate firm-level heterogeneity with data on exports of each firm in Sweden in the manufacturing sectors (NACE15-37), observed over the period 1998-2004. The description is based on the variety of firms' exports. We first describe the data and describe the measurement of varieties at the level of individual firms, where the exported products of each firm are tabulated across destination countries. Then, we go on to depict the distribution of firms according to the size of their variety stock and the invariance of this distribution over time. Finally, we illustrate the frequency of entry and exit of varieties by firms and how the distribution of entry and exit is related to the distribution the variety stock.

2.1 Data description and definition of variables

We make use of data maintained by Statistics Sweden which include information on firms' export activities on a yearly basis between 1998 and 2004.[†] Firms are defined as legal entities, and the data provide information on each firms' exports to each country and year, measured in terms of value (SEK) and volume (kilogram). An advantage of using export data is that they contain detailed information on the supply pattern of each firm. The export flows of each firm are tabulated over product varieties and markets. A product variety is here defined as a product classification code at the 6-digit level according to the CN classification scheme.[‡] At this level of aggregation there are over 8 000 product codes. The structure of the data makes it possible to identify the number of product varieties, number of destination countries as well as the number of unique combinations of product varieties and destination markets of each firm over time.

To illustrate firm-level heterogeneity we derive a measure of the variety of firms' exports, i.e. a measure based on observable features of firms' actual sales of different products on different markets. Let i denote a product variety, i.e. product code at the 6-digit level, and j denote a destination market (country). A firm may then be characterized by a destination-variety matrix $\Omega = \{\omega_{ij}\}$, where $\omega_{ij} = 1$ if variety i is sold on market j and 0 otherwise. Our measure of a firm k's export variety is given by the summation $n^k = \sum \omega_{ij}^k$, which is the number of distinct combinations of product varieties and destination markets of each firm k. In every year, n^k provides a complete measure of the variety of firms' export flows. Henceforth, we refer to n^k as the *variety stock* of a firm. Observe that non-exporting firms are defined by the condition $n^k = 0$, whereas all exporting firms have $n^k \geq 1$.

An implicit assumption behind the described measure of a firm's variety stock is that a product variety exported to market j is differentiated from the same product variety exported to a different market l. This assumption may be viewed as a firm-level dual to the well-known 'Armington

[†] Detailed description of the data can be found in Andersson et al. (2008) and Andersson and Johansson (2008).

[‡] CN is an acronym for the Combined Nomenclature. When declared to customs in the EU, goods must generally be classified according to the CN. The CN is a method for designating goods and merchandise which was established to meet, at one and the same time, the requirements both of the Common Customs Tariff and of the external trade statistics of the Community. The CN is also used in intra-Community trade statistics. The CN is comprised of the Harmonized System (HS) nomenclature with further Community subdivisions. The Harmonized system is run by the World Customs Organization (WCO).

assumption' in Armington (1969), which states that products are distinguished by place of production. In the variety stock measure applied here, the individual varieties are distinguished by place of consumption in combination with a product classification code (6-digit CN). The same 6-digit CN exported to two different countries counts as two destination varieties. In the sequel, we will illustrate the level of heterogeneity across Swedish manufacturing firms in terms of the variety stock.

In order to illustrate dynamics in firms' supply pattern over time, we also identify the entry and exit of destination varieties in firms' variety between time periods. That is, we will identify changes in the destination-variety matrix $\Omega = \{\omega_{ij}\}$. Entry of a destination variety occurs when ω_{ij} = 0 in period *t*, but ω_{ij} = 1 in period ($t+\tau$).[§] Similarly, exit occurs whenever ω_{ij} = 1 in period *t*, but ω_{ij} = 0 in period ($t+\tau$). In the analysis we have t = 1998 and ($t+\tau$) = 2004. Our illustration of firm-level heterogeneity in the next section is thus based on three different measures:

(i) Variety stock and its change over time = number of destination varieties

(ii) New varieties = new varieties of the firm in 2004 compared to 1998

(iii) Exit = varieties that vanish between 1998 and 2004

The variety stock measure allows us to depict the heterogeneity of firms in terms of observable features of firms' actual sales of different products on different markets in each time period. The identification of the entry and exit of destination varieties over the period 1998-2004 allow us to assess dynamics in the firm's supply pattern over time, and this dynamics may be compared the dynamics of a firm's variety stock.

2.2 Distribution of firms across variety stock and the dynamics of entry and exit of varieties

Figure 1 present the distribution of firms across different levels of the variety stock in 2004. The horizontal axis measure the extent of the variety stock in logs. The vertical axis measures the log of the number of firms. Each dot in figure then shows the number of firms associated with given variety stock according to the horizontal axis.

[§] Note that a firm exporting product variety *i* to a new market *m* will expand its variety stock. The same takes place if the firms initiate export of a new product variety *s* to a market *j* which it already exports other product varieties to. Hence no distinction is made between a new product variety (CN6-code) to a destination country already served, or a new destination country of a product variety already in a firm's export supply schedule.

The pattern in Figure 1 relates to the well-known rank-size distribution and to previous observation of skewed distributions of firm size (Ijiri and Simon 1977, Axtell 2001 and 2006). The picture reveals that the number of firms falls of monotonically as the variety stock increases. The distribution of firms over variety stocks is skewed: in terms of the magnitude of their variety stock, firms are truly heterogeneous. Many firms are associated with only one destination variety, i.e. one product variety sold on one market and a smaller number of firms supply many product varieties to different markets. There is hence co-existence of large multi-variety firms and small single variety firms.

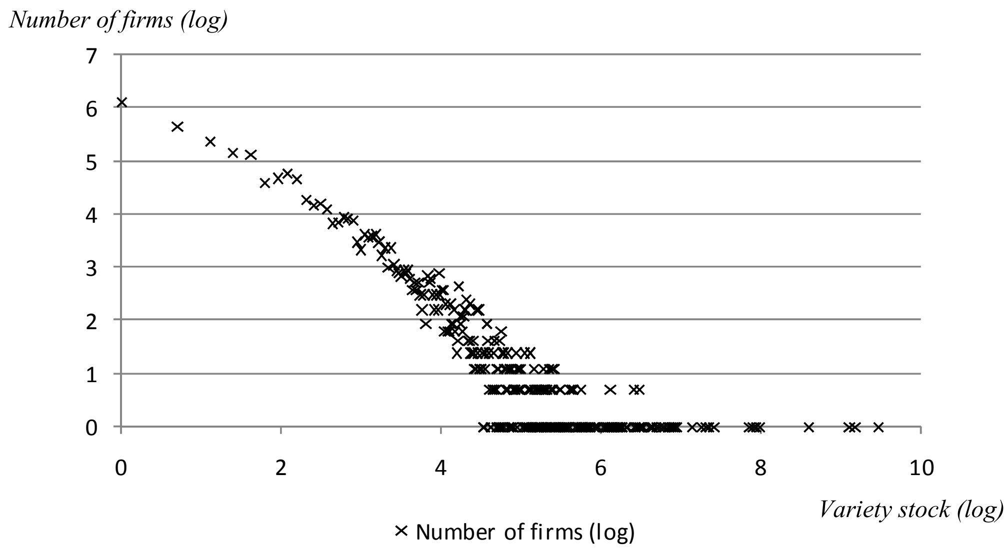

Figure 1. The distribution of varieties across firms among export firms in 2004.

The distribution depicted by Figure 1 is for the year 2004, but remains rather invariant over time when looking at the period 1998-2004. Firms tend to remain within or in the neighborhood of the same magnitude of the variety stock over time. We illustrate this by studying the 'diagonal bias' of a transition matrix depicting firms' switching between different states in terms of the magnitude of their variety stock. Let q_{kk} denote the probability that a firm in state *k* will remain in the same state over a time interval τ, i.e. 1998-2004. Furthermore, let q_{kk-1} and q_{kk+1} denote the probability that a firm in state *k* switches to the nearest state below and above the initial state, respectively, during the same time interval. For each initial state *k*, q_{kk} corresponds to the diagonal elements of a transition matrix, whereas q_{kk-1} and q_{kk+1} correspond to one step left and one step right, respectively, off the diagonal. The two measures q_{kk} and $\hat{q}_k = q_{kk} + q_{kk-1} + q_{kk+1}$ then provide a description of the 'diagonal bias' of the observed state transitions of firms over time.

Table 1 presents the values of q_{kk} and $\hat{q}_k$ based on observations of how firms switch states in terms of variety stock over the seven-year period 1998-2004. For each year, we construct 15 states reflecting different orders of magnitude of the variety stock, where state 1 comprise firms with zero varieties, i.e. non-exporters, and state 15 those firms with a large number of varieties ($\geq$ 135). The principle of equal percentiles is applied in the groupings, though state 1 contains a relatively large set of firms given the nature of the data. We then calculate state transition probabilities by observing the state of each firm in each and every year.

Table 1. State Transition Probabilities 1998-2004

State in period t	***Average yearly share of firms in group (%)***	q_{kk} **(%)**	$\hat{q}_k = q_{kk} + q_{kk-1} + q_{kk+1}$ **(%)**
Non exporter (0 export varieties)	26.4	81.8	92.7
1 variety	8.3	31.6	81.8
2 varieties	5.7	23.2	63.2
3 varieties	4.3	17.6	60.7
4-5 varieties	6.7	29.5	61.5
6-7 varieties	4.7	23.6	64.7
8-9 varieties	4.1	24.2	63.3
10-12 varieties	4.5	27.1	67.1
13-16 varieties	4.7	33.1	75.8
17-22 varieties	5.0	39.4	82.5
23-31 varieties	5.2	46.4	88.2
32-45 varieties	4.8	51.8	93.4
46-66 varieties	4.7	58.8	96.1
67-135 varieties	5.7	76.7	98.3
$\geq$ 135 varieties	5.3	92.1	99.5

It is evident from the table that the transition probability matrix has a diagonal bias. For all states in period *t*, $\hat{q}_k$ is over 60 %. Hence, firms tend to remain within or in the neighborhood of the same state over time. This reflects the invariance in the structure illustrated by Figure 1 over time; firm-level heterogeneity in variety stock is persistent. However, despite the described invariance in structures, there is significant gross dynamics in terms of entry and exit of destination varieties across firms. The elements in firms' destination-variety matrix Ω, ω_{ij}, switch state, i.e. from 0 to 1 and vice versa, frequently between periods, though the variety stock of a firm *k*, $n^k = \sum \omega_{ij}^k$, tends to be invariant.

Figure 2 illustrate the distribution of new and vanishing destination varieties across firms that export persistently over the period 1998-2004 and compares them with the distribution of the variety stock in 1998.[**]

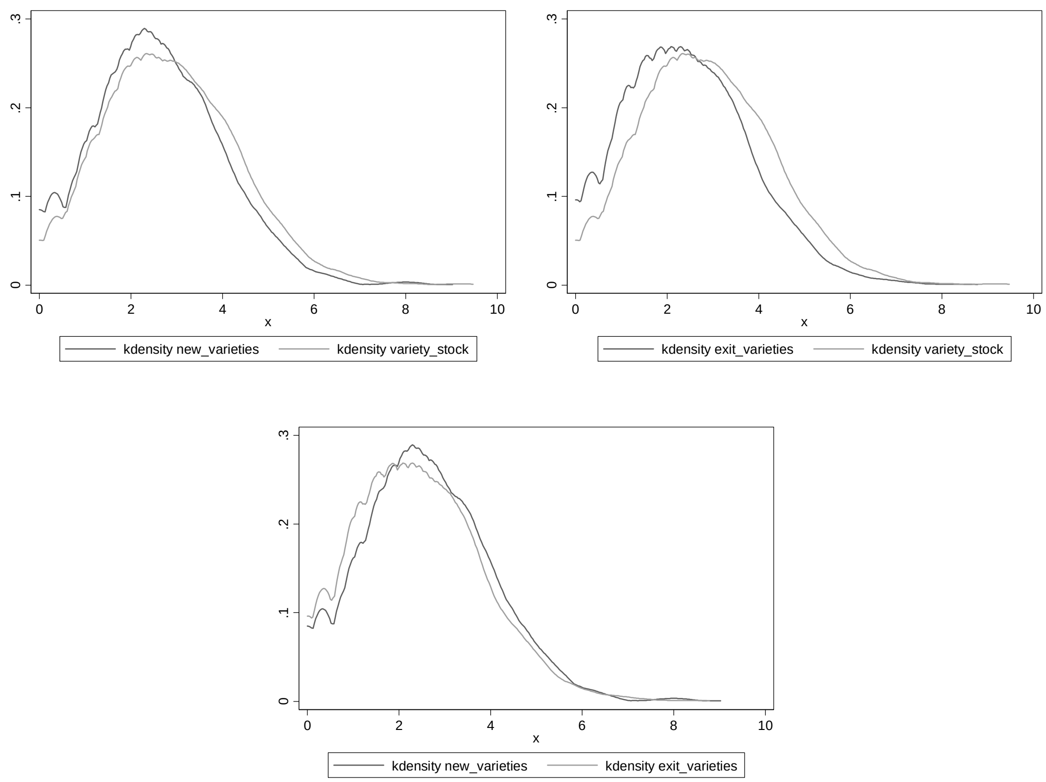

Figure 2. Kernel densities of (log) exiting, (log) new destination varieties and (log) variety stock.

It is evident from the figures that the distribution of new and vanishing destination varieties of firms that export persistently 1998-2004 show similar properties as the distribution of their variety stock in the base year 1998. The similar distribution between variety stock, new varieties and vanishing varieties illustrate that the invariance in variety stock conceals significant gross dynamics of entry and exit of destination varieties. However, the micro-dynamics of the supply pattern of individual firms is such that the overall distribution remain intact over time.

Table 2 shows that the micro-dynamics of firms' export supply pattern is economically significant. About 20 percent of the export value of exporting firms in 2004 was generated by destination varieties that were not included in the firms supply schedule in 1998. The firms considered were also exporters in 1998, such that a fifth of their total export value in 2004 was generated by new destination varieties

[**] We are here interested in changes the supply pattern of firms over time as evidenced by the composition of their exports and therefore exclude firms that exports temporarily over the period 1998-2004.

compared to 1998. Looking at the unweighed average (average of the individual firms' share), the figure amounts to 40 percent, but the distribution of fraction of export value generated by new destination varieties is skewed. The median fraction is half of the average and the fraction ranges from 0 to 100 percent. Both the weighed and the unweighed figure however show that the entry and exit of destination varieties is indeed economically significant.

Table 2. Economic significance of entry and exit of destination varieties.

Fraction of export value in 2004 generated by new varieties (fraction of total, sum across all persistent exporters)	0.21
Fraction of export value in 2004 generated by new varieties (based individual firms' fraction)	
Mean	0.40
Median	0.25
Std. Deviation	0.37
Min	0.00
Max	1.00

To summarize, we have illustrated that:

- there is strong heterogeneity in the supply pattern of firms, both in terms of the number of products and markets they serve. The distribution of firms across variety stock (extent of destination varieties) resembles the established skewed distributions of firm size.
- there is persistence in the distribution of firms across variety stock. Transition probability matrix between different states (variety stocks) is diagonal heavy, such that firms tend remain in (or be in the neighborhood of) the same state in terms of variety stock over time.
- Despite diagonal bias of transition matrix, there is significant gross dynamics of entry and exit of destination varieties. Even though the magnitude of an individual firm's variety stock remain invariant over time, the components of the variety stock (the destination varieties) are renewed over time through entry and exit of destination varieties.

3. THE MODEL

How can we understand and model the observed persistent heterogeneity across firms and its associated micro-dynamics? We will here outline a parsimonious model that has the ambition to capture the picture of heterogeneous firms described in the previous section, as well as accommodate the simultaneous exit and entry of destination varieties in firms' supply pattern.

The basic premise of the model is that each product variety of a firm is based on an innovation idea. Innovation ideas are assumed to arrive to existing as well as potential entrepreneurs, and these ideas are materialized in the form of new product varieties that are sold on one or several destination markets. Obsolete varieties are assumed to exit from the market according to a stochastic process, which means that taste for variety has a temporal dimension.

We assume both scale economies of firms and path-dependence, such that the arrival rate of innovation ideas to an individual firm is a function of each firm's stock of varieties and cumulated experiences at every given point in time. Conservation mechanisms include development of demand and exit of established varieties. The state-dependent Poisson processes are individual to each firm, and can be transformed into a Markov process, depicting the transition of firms between different states. Under given conditions this process has a stationary distribution where firms of different size co-exist, and destination varieties enter and exit simultaneously.

3.1 Cost of Production and Destination Markets

We consider an economy with many countries (destination markets), a set of product groups, where each such group contains differentiated varieties. For each such product group, we assume that in every destination market the customers have preferences implying taste for variety. In the model of this economy we examine individual firms selling their varieties to some or all destination markets. All supply firms in the model are located in one country (Sweden), and our product-variety specific demand functions makes this theoretically feasible.

The demand structure in the domestic as well as in each of the destination markets is such that every product variety is supplied under conditions of monopolistic competition, and market solutions are identified for a sequence of points in time. A typical sales flow is signified by x_{ijk}, where i refers to a particular variety, j to a specific destination market, and k to an individual firm. The flow x_{ijk} will exist if firm k has made the destination-variety innovation, combining the variety i and the market j. This is in accordance with the empirical measure of destination varieties introduced in Section 2. The prerequisite for making an innovation is that the firm receives an innovation idea, where ideas are generated by a stochastic process which is specific for each firm. Moreover, an innovation is always a combination, (i, j), specifying both variety and market, i.e. a destination-variety combination.[††] Once a variety innovation is made, the firm can over time add new markets for this variety – given that such market ideas emerge. It may also add new varieties to existing markets. In order to simplify the formalism, we assume that a firm exporting a variety i always supplies the same variety to the domestic market, $j = d$.

[††] This is in accordance with the Schumpeter observation that innovations include both a variety and a market novelty, recognizing that an innovation must find its way to customers (Schumpeter 1934).

When a viable innovation idea, (i, j) arrives to a firm (or a potential firm), it has to make a product-development investment, $\hat{G}_i$ if the variety i is new to the firm and a market investment, $\hat{H}_j$, if market j is also new, except for the domestic market, where we follow the tradition from New Economic Geography and assume $H_d = 0$ (cf. Krugman 1990). For a given variety i, the investment $\hat{G}_i$ can be relied upon for all destination links that over time may develop on the basis of a sequence of market innovation ideas. In a similar way, a given destination-link investment can be utilized for several different varieties. When this applies, a multi-variety and multi-destination firm will benefit from economies of scope.

The investments $\hat{G}_i$ and $\hat{H}_j$ can be transformed to costs per unit of time, denoted by G_i and H_j. In addition, we consider that the firm has a firm-specific fixed cost per unit of time, denoted by F. Given this, the following assumption is introduced:

> (A.1) Every firm has three types of capital: F-capital representing the firm-specific asset, a G-capital for each variety, and a H_j-capital for each destination link j, while observing that $H_d = 0$. The variable cost of each firm is v per unit output. A firm is characterized by its destination-variety matrix $\Omega = \{\omega_{ij}\}$ defined below:

$$\omega_{ij} = \begin{cases} 1 & \text{if variety } i \text{ is sold on market } j \\ 0 & \text{otherwise} \end{cases}$$

The Ω-matrix provides a complete description of a firm's collection of destination varieties and was introduced empirically in Section 2. From the matrix we can also construct the two summary variables. The number of specific varieties supplied by a firm is given by $\sum_i \omega_{iJ}$, where:

$$\omega_{iJ} = \begin{cases} 1 & \text{if } \sum_j \omega_{ij} \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

In a similar way the number of destination links is given by the summation $\sum_i \omega_{Ij}$, where:

$$\omega_{Ij} = \begin{cases} 1 & \text{if } \sum_i \omega_{ij} \geq 1 \text{ for } j \neq d \\ 0 & \text{otherwise} \end{cases}$$

For a specific firm, k, we may calculate $G^k = \sum_i \omega_{iJk}$ denoting the number of varieties of firm k, and $H^k = \sum_{j \neq d} \omega_{Ijk}$ denoting the number of distinct links of firm k.

Consider now the potential presence of scope economies, and let $\#(x_{ijk})$ denote the number of flows for which $x_{ijk} > 0$, as specified by the firm's Ω-matrix. Examining the cost expression for firm *k*, $C_k = v\sum_i \sum_j x_{ijk} + G^k G + H^k H + F$, we can easily see that $\#(x_{ijk}) > G^k + H^k$, which means that certain different varieties use the same destination link, and one variety can be sold via several different links. This shows that scope economies are present. This fact plays in the present analysis only one important role: it motivates why monopolistic-competition firms are willing to supply many varieties.

In (A.1) it is assumed that the variable cost, *v*, does not vary between firms. Hence, innovations do not change variable costs. Instead an individual firm can improve its productivity and gross profit by means of innovations that increase the firm's economies of scope..

3.2 Demand for Varieties and Prices in each Destination Market

We introduce a demand function for each product variety, based on the assumption that customers has CES preference functions. Assuming that customers optimize and that the number of varieties in the selected product group is large enough to make the income effect negligible (Dixit and Stiglitz 1977), we can derive the following demand function:

$$x_{ij} = \alpha_j p_i^{-\theta} M_j \tag{1}$$

where *i* denotes a given variety and *j* a given destination market, and where the parameter $\theta = 1/(1-\phi) > 1$ represents the price elasticity and where α_j applies for all product varieties $i \in I_j$ in each destination market *j*. The value of α_j reflects the market structure and shrinks as the number of varieties increases. If each supplier perceives the type of demand function given in (1), monopolistic competition between varieties applies and then we get $\alpha_j = P_j^{\theta-1}$, where $P_j \equiv \left(\sum_i p_{ij}^{1-\theta}\right)^{\frac{1}{1-\theta}}$ is the ideal price index (Dixit and Stiglitz 1977).

We assume that each firm that sells to customers in market *j* perceives P_j and hence α_j as given. This implies that such a firm selects an optimal price by maximizing the gross profit associated with the flow x_{ij}, which means that the firm maximizes $(p_{ij} - v)x_{ij}$. As shown in Andersson and Johansson (2008), the firm will select the price:

$$p_{ij} = p^o = v\theta/(\theta - 1) \quad (2)$$

Thus, in each destination every variety is sold at the price p^o, which implies that $x_{ij} = \alpha_j (p_i^o)^{-\theta} M_j$, where the size of the coefficient α_j depends on the number of varieties supplied, and at each point in time the number of varieties is given by past innovations. The price in (2) is only viable if the net profit is non-negative. For an L1-firm located in market *j* and supplying just one variety at home, the net profit can be expressed by $V_{L1} = (p^o - v)x_{id} - F - G$, and if $V_{L1} \geq 0$ all foreign firms exporting to market *j* will have a non-negative net profit for each variety flow, given that $F + G > H$. This follows since, contrary to L1-firms, all foreign firms have at least two markets – one export and one domestic.

How can we keep track of which varieties are supplied to market *j*, and which varieties are not? For any given point in time, $I_j^o \subset I_j$ denotes those varieties that are currently supplied. Suppose now that $V_{L1} = 0$ in market *j*. Then we can conclude that the market is saturated in the sense that:

$$p^o x^o n_j^o = M_j \quad (3)$$

where x^o is the value of sales that makes $V_{L1} = (p^o - v)x^o - F - G = 0$, and where $n_j^o = M_j / p^o x^o$. Then, what applies if I_j^o contains only $n_j < n_j^o$ varieties? In such a case then demand function tells us that each delivery flow satisfies $x_{ij} > x^o$, where $\sum_{i \in I_j^o} p_{ij} x_{ij} = M_j$. We may think of $n_j (x_{ij} - x_j^o)$ as a gap in the market, implying that new varieties can be introduced whenever $n_j (x_{ij} - x_j^o) > x^o$. The exit process of the model helps to maintain a sufficient gap for the entry dynamics.

3.3 Exit of Varieties and Arrival of Ideas

Consider the number of varieties, n_j, supplied to market *j* at a given point in time. This number is constrained by two factors: (i) the size of M_j and (ii) the historically given number of varieties, "enumerated" in the set I_j^o, that have been developed by innovating firms selling to market *j*. What will happen with the established varieties as time goes by? We assume that the taste for variety has a temporal dimension, such that for each market there is a positive probability for any variety of becoming obsolete so that it disappears from the set $I_j^o \subset I_j$. Formally, we assume:

(A.2) Over time, a stochastic process generates exit of varieties from each index set I_j^o , where the exit of variety $i \in I_j^o$ reflects that this variety has become obsolete in view of the customers in market *j*.

This assumption puts the CES preference structure in a new perspective. The assumption plays an important role by augmenting the options of firms to make variety innovations. In addition, the relevance of the assumption will be assessed empirically.

How do innovation ideas arrive to established and potential firms? In the model we assume that the arrival is governed by a stochastic process of the following nature:

(A.3) Innovation ideas arrive according to a Poisson process, and this process is specific for each established and each potential firm, in the sense that the rate of arrival of ideas can differ between firms and can evolve over time. An innovation idea is turned into a destination variety (an innovation) by the firm, given that the firm's net profit is not reduced by the decision to innovate.

Introducing (A.3) provides a way to capture systems properties of all sorts of complex phenomena that influence the creation of new ideas by entrepreneurs in the set of existing and potential firms. In this way we use (A.3) as a tool to systematically investigate firm attributes and other factors that can influence the arrival rate of ideas. The most obvious way to make the arrival rate of firm *k*, λ_k , state dependent is to assume that $\lambda_k = \lambda(n_k)$, which assumes that the firm's innovation history is reflected by its stock of active destination varieties, n_k , and that the size of this stock influences the speed of arrival in a positive way. This is similar to the model developed by Klette and Kortum (2004), in which a firm of any size adds new products by innovating, but in any period its likelihood of success depends on its knowledge capital accumulated through past product innovations.

3.4 Stationary Properties of the Supply Pattern

The assumption is that at any given point in time ideas arrive according to a firm-specific Poisson process. Those ideas can materialize into new varieties if the specific market allows the introduction of a new variety. The latter is influenced by the Poisson exit process. The objective here is to depict all these individual processes as parts of a Markov process. We do this be specifying transition rates, where the rate q_{kl} combines the probability of variety entry and exit, as given by the Poisson arrival and exit process relevant for *k*-firms. On the basis of Appendix 1 we assume:

(A.5) Let $q_{kl}(t, t+\tau)$ denote the probability that a firm in state *k* moves to state *l* in the time interval τ. If we let $\tau \to 0$ we obtain a transition matrix $Q = \{q_{kl}\}$, where *k*, *l* = 0,1,....,*n*, and where $q_{kk} = -\sum_{l \neq k} q_{kl}$. We assume that at each time *t* there is a vector $f(t) = [f_o(t), f_1(t), ... f_k(t), ...]$ describing the share of firms in each state, where $f_0(t)$ is the share of potential firms. The change of the supply structure is described by $df(t)/dt = f(t)Q$.

The *Q*-matrix summarizes the dynamics of the model introduced in previous sections. The matrix will depict the dynamics in an exact way as long as each market has a small gap that allows innovation ideas to materialize as new destination varieties. We will refer to this condition by saying that we study the process of entry and exit when every market is in the neighborhood of its equilibrium. Given that, the following conclusion can be drawn:

Proposition: Assume that the Q-matrix is irreducible. Then the assumption in (A.4) implies that the Markov process defined by the *Q*-matrix has a stationary solution $f^* = (f_0^*, ..., f_k^*, ..)$ such that $f^*Q = 0$, which means that process has stationary distribution of firms with different numbers of destination varieties (Andersson and.Johansson 2009).

Our final reflection is on firms that have to be closed down. When that happens it can be represented by a small exogenous shock of the distribution $f^* = (f_0^*, ..., f_k^*, ..)$, resulting in a moderately modified stationary solution, which follows from the proposition (Clegg 2008). Given that the *Q*-matrix is irreducible to satisfy ergodicity properties, the equilibrium distribution of firms exists and can be obtained as $f^* = \lim_{t \to \infty} f(t)$, satisfying $f^*Q = 0$. The introduced transition rate q_{kl} reflects the rate at which a firm with *k* destination varieties experiences the change $l - k = a_k - e_k$, where k is the firm's current number of destination varieties, a_k is the number of destination-variety ideas that arrive to the firm, and where e_k is the number of destination-varieties that exit from the current stock of such varieties. All these transition rates can be derived from Poisson probabilities related to each individual firm.

4. SIMULATIONS

In this section we aim to assess whether the model outlined in the previous section with state-dependent firm-specific Poission processes is sufficient to capture the main elements of the heterogeneous distribution of firms and the micro-dynamics of their supply pattern. We apply simulations to describe how well the model derived in the previous section describe the development of entry and exit of products over time. The purpose is to examine whether we can observe the same dynamic pattern by generating the development of entry and exit of products using the assumptions of the model as we can observe for Swedish data.

The main questions we want to answer is first whether an initially skewed distribution remains skewed over time with state-dependent. Secondly, we want to see if an initially homogenous distribution becomes skewed over time. Thirdly, we want to investigate whether the dynamics are stable, or if there are 'critical values' for key parameters at which the system explodes through bifurcation.

4.1 Simulation design

The assumption of the model is that new ideas arrive according to a Poisson process where the arrival rate of innovation ideas is state-dependent, such that it depends on the number of innovation ideas the firm has already materialized. In this way, the arrival rates are firm-specific.

We use both a "linear" entry where $\lambda = \exp\left(\alpha n_i\right)$ and a "S-shaped" entry where $\lambda = \exp\left(\alpha n_i + 100\alpha n_i^2 - 10000\alpha n_i^3\right)$ [‡‡]. The difference between "linear" and "S-shaped" entry is that the intensity of the Poisson process grows more rapidly for the "S-shaped" entry for low n than the "linear" entry. Besides the shape of the entry we also vary the value of the α-parameter. The entry rate is changed by altering the α parameters to find 'critical values' at which the system explodes. The exit rate of products is also assumed to follow a Poisson process with an intensity equal to $\mu = \exp\left(\beta \sum_i^N n_i\right)$. Thus the exit rate is not firm-specific and it depends in a positive way on the number of products in the system. The β parameter is held constant and it equals 0.00005 throughout the experiment.

When it comes to initial distributions of firms across variety stock, we use a heterogeneous distribution where most firms only have a few products and the largest firm have 300 products. Then we also use a homogeneous distribution where there are an equal amount of firms that have one product up to ten products. Another factor we vary is the number of time periods where we use t+10

[‡‡] The intensity is growing up to 1000 products since the intensity of the S-shaped distribution starts to decrease for n>1000.

and t+100. By increasing the number of time periods we are able to investigate the robustness of the results since we are able to see if the dynamic pattern is still holding over a longer time period.

4.2 Results

Figure 3 shows the distribution looks at time period t and time period t+10 when we start with a heterogeneous and a homogenous initial distribution for $\alpha = 0.005$. One can see that the skew distribution remains skewed and that the homogenous distribution becomes skewed over time. Thus, the outlined dynamics of the model in Section 3 with state-dependent Poisson processes are not only able to replicate the stylized fact that the skewed distribution remains skewed over time, it is also able generate a skewed distribution from a homogeneous one.

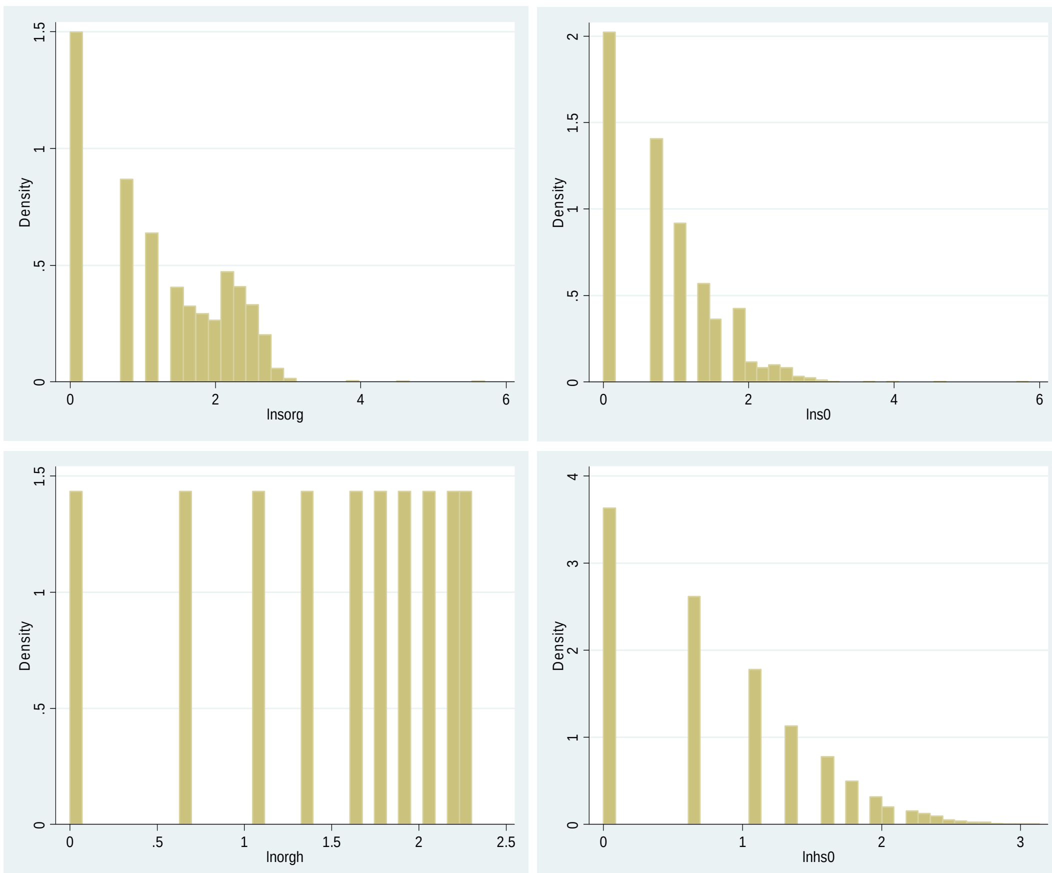

Figure 3. The distribution of varieties across firms for simulated data. Left figure in upper and lower part is the initial distribution. The right figure shows the distribution after 10 periods.

Table 3 shows the estimated the transition probabilities for Swedish data and for simulated data where the initial distribution is skewed and $\alpha = 0.005$. We use the same grouping of firms into different states of variety stock and examine whether the transition matrices for simulated data and actual data show a similar diagonal bias.

For both Swedish data and simulated data the tail probabilities are heavy. Thus for both Swedish data and for simulated data the probability is larger for companies that do not export to continue not to export and for large companies to continue having a large variety of products than the probability that firms with only a few products continue to produce only a few products. The results using *t*+100 are very similar to the result for *t*+10 which shows that the results we obtain from the simulations are robust. However, due to the similarity the results for *t+100* is omitted but the results are available from the authors upon request.

Table 3. State transition probabilities for simulated data

	Simulated data		**Actual Swedish data**	
State in period t	q_{ii} **(%)**	$q_i = q_{ii} + q_{ii-1} + q_{ii+1}$**(%)**	q_{ii} **(%)**	$q_i = q_{ii} + q_{ii-1} + q_{ii+1}$**(%)**
Non exporter	89.12	97.42	81.82	92.72
1 variety	27.81	90.36	31.57	81.77
2 varieties	27.42	75.59	23.19	71.59
3-4 varieties	47.55	85.16	32.34	74.86
5-8 varieties	69.41	96.75	44.69	82.48
9-12 varieties	69.38	99.62	35.78	81.86
13-16 varieties	67.49	99.63	33.08	89.94
>17 varieties	81.14	99.66	92.88	97.24

In Figure 4 the number of firms in the system after 10 periods for different values of α is shown. The difference between "linear" and "S-shaped" entry is very small and we therefore only show results for the "S-shaped" entry. Results for the "linear" entry is, however, available from the authors upon request. The figure is intended to show whether there are critical values for α at which the system collapses.

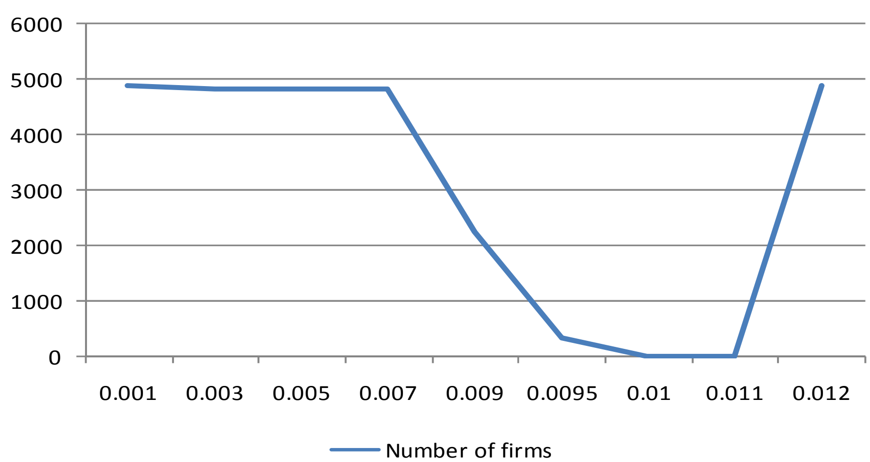

Figure 4. Number of firms for different values of the α -parameter

As can be seen from the figure, there are parameter values at which the system collapses. The number of firms is stable between 0.001 and 0.007. Then it starts to decrease and when α is equal to 0.01 there only exists one firm. When α is greater than 0.01 the only remaining firm loses all of its products. This happens since we have limited the intensity of the entry process while the exit process is unlimited.

5. SUMMARY AND CONCLUSIONS

Firm-level data reveals strong heterogeneity across firms. We made us of Swedish data depicting manufacturing firms' sales of different products on different markets over time and illustrated the co-existence of large multi-variety firms and small single-variety firms. In particular we showed that:

- There is strong heterogeneity in the supply pattern of firms, both in terms of the number of products and markets they serve. The distribution of firms across variety stock (extent of destination varieties) resembles the established skewed distribution of firm size

- There is persistence in the distribution of firms across variety stock. Transition probability matrix between different states (variety stocks) is diagonal heavy.

- Despite diagonal bias of transition matrix, there is significant gross dynamics of entry and exit of destination varieties. Even though the magnitude of an individual firm's variety stock remain invariant over time, the components of the variety stock (the destination varieties) are renewed over time.

We outlined a parsimonious model can capture the picture of heterogeneous firms and accommodate the simultaneous exit and entry of destination varieties in firms' supply pattern. The basic premise of the model is that each variety of a firm is based on an innovation idea and that innovation ideas arrive to existing as well as potential entrepreneurs. An important element of the proposed model is state-dependent Poisson processes for the arrival rate of innovation ideas, such that the arrival rate is firm-specific and partly determined by each firm's stock of varieties and cumulated experiences at every given point in time. The state-dependent Poisson processes can be transformed into a Markov process, depicting the transition of firms between different states.

The simulation study showed that this simple formulation of innovation at the firm-level is capable of capturing the 'big picture' of the observed heterogeneity across firms and the associated micro-dynamics. The described innovation processes at the firm-level can retain the persistence skewed distribution of firms, in particular the persistent coexistence of single- and multi-variety firms. It also accommodates the dynamics of firms' supply pattern that is observed on actual data. Moreover, a skewed distribution compatible with the literature on the size distribution of firms may also be generated form an initially homogeneous one.

In summary, the simple model outlined in the paper as well as its associated evaluation (or innovation) dynamics driven by state-dependent Poisson-processes is consistent with the Swedish data as well as a large part of the stylized facts on the size distribution of firms as reported in e.g. Klette and Kortum (2004). The firm-level innovation process modeled in the paper leads to persistent heterogeneity across firms.

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APPENDIX 1: Transforming Poisson processes into a Markov Process

Consider that at a given point in time, t, there is a vector $f(t)=[f_0(t), f_1(t),...]$ of probabilities or shares such that $\sum_k f_k(t)=1$, where $f_n(t)$ denotes the share of firms with n distinct (i, j)-combinations. Following Clegg (2008) we shall study the Poisson arrival of innovation ideas and exit of (i, j)-combinations with the help of a continuous-time Markov chain, where σ_{kl} denotes the flow rate between state k and l. With small discrete time steps, δt, the transition pattern can be depicted by

$$P(\delta t)=\begin{bmatrix} 1-\sigma_{00}\delta t & \sigma_{01}\delta t & ... & \\ \sigma_{10}\delta t & 1\text{-}\sigma_{11}\delta t & ... & \\ \sigma_{20}\delta t & \sigma_{21}\delta t & 1\text{-}\sigma_{22}\delta t & ... \\ - & & & \end{bmatrix} \tag{A1.1}$$

It is not meaningful to take the limit for $\delta t \to 0$. Instead we introduce $q_{kl}(t, t+\tau)$ to denote the probability of a transition from state k to state l. Since there is no temporal memory (homogenous time), we may construct the following transition rates:

$$q_{kl} = \lim_{\tau \to 0} q_{kl}(\tau)/\tau \tag{A1.2}$$

The transitions depicted are like a Poisson process, and then it is possible to write:

$$f_k(t+\delta t) = f_k(t) - \sum_{l \neq k} f_k(t)\sigma_{kl}\delta t + \sum_{l \neq k} f_l(t)\sigma_{lk}\delta t + \mathrm{O}(\delta t) \tag{A1.3}$$

Differentiating with respect to time yields the following result :

$$\partial f_k(t)/\partial t = -\sum_{l \neq k} f_k(t)\sigma_{kl} + \sum_{l \neq k} f_l(t)\sigma_{lk} \tag{A1.4}$$

Now, formula (A1.3) can be replicated in terms of $q_{kl}(t)$, which yields:

$$f_k(t+\delta t) = f_k(t) - \sum_{l \neq k} f_k(t) q_{kl}(\delta t) + \sum_{l \neq k} f_l(t) q_{lk}(\delta t)$$
(A1.5)

From this we can replicate (A1.4) to obtain $\partial f_k(t)/\partial t = -\sum_{l \neq k} f_k(t) q_{kl} + \sum_{l \neq k} f_l(t) q_{lk}$. Then, to make the two processes consistent, we assume that q-coefficients and σ-coefficients correspond so that $q_{kl} = \sigma_{kl}$ for $k \neq l$ and that $q_{kk} = -\sum_{l \neq k} q_{kl}$. These *q*-coefficients can be arranged in a *Q*-matrix, $Q = \{q_{kl}\}$, for which we have that $df(t)/dt = f(t)Q$.

Given that the *Q*-matrix is irreducible to satisfy ergodicity properties, the equilibrium distribution of firms exists and can be obtained as $f^* = \lim_{t \to \infty} f(t)$, satisfying $f^* Q = 0$. The introduced transition rate q_{kl} reflects the rate at which a firm with *k* destination varieties experiences the change $l - k = a_k - e_k$, where k is the firm's current number of destination varieties, a_k is the number of destination-variety ideas that arrive to the firm, and where e_k is the number of destination-varieties that exit from the current stock of such varieties. All these transition rates can be derived from Poisson probabilities related to each individual firm.