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Testing for Panel Unit Roots under General Cross-Sectional Dependence

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Abstract: In this paper we generalize four tests of multivariate linear hypothesis to panel data unit root testing. The test statistics are invariant to certain linear transformations of data and therefore simulated critical values may conveniently be used. It is demonstrated that all four tests remains well behaved in cases of where there are heterogeneous alternatives and cross-correlations between marginal variables. A Monte Carlo simulation is included to compare and contrast the tests with two well-established ones.

1. Introduction

In order to assess stationarity of a process, it is common to express it by means of a statistical model. Traditionally this has been done using a first-order autoregression model where the stationarity is assessed through tests of the linear restrictions on the autoregression parameter – see, for example, Fuller (1996), Dickey and Fuller (1979), Said and Dickey (1984) and Ng and Perron (1995). These tests, however, are designed for univariate processes, and it is by no means obvious how to extend a univariate test to a multivariate setting of, say, N variables such as in a panel data context, though some tests of different origin have been considered in the literature. Breitung and Meyer (1994) suggested a panel data unit root test valid for a fixed number of observations T and $N \to \infty$. Apart from the assumption of increasing N, their test, as well as the test proposed by Levin, Lin and Chu (2002), assumes a common scalar parameter, which does not allow some marginal series to be unit roots and others not. Im, Pesaran and Shin (2003) relaxed the assumption of a common parameter and proposed a test based on averaging the test statistics from the N cross-section units. Another test which allows for heterogeneous parameters over the N units was introduced by Maddala (1977), who used Fishers p-value statistic $-2\sum_{i=1}^{N} \log P_i$ where P_i is the p-value of any univariate test of the ith marginal process. This, as well as the other panel data tests above, does not allow for cross-sectional correlations, which in turn appear to be present in most cases met in practice. In such cases, Maddala and Wu (1999) suggest using 'bootstrap' methods to obtain fair, critical values of Fisher's test. A somewhat different test was proposed by Moon and Perron (2004), who assumed a factor structure of the cross-covariances and suggested pooling defactored variables to construct a unit root test. Other factor-based methods for cross-correlated data have been proposed by Phillips and Sul (2003), Bai and Ng (2004), Choi (2002) and Pesaran (2003).

In this paper, we will take a different approach to panel data unit root testing, which simultaneously allows for heterogeneous unit roots – in the sense that some series may be stationary and others not – and cross-correlations of the residuals without assuming a factor structure. We specify the panel unit root problem in the form of a multidimensional restriction on a linear, but (possibly) non-stationary, vector process and four invariant statistics for testing the hypothesis. As each of the proposed statistics is invariant to linear transformations, the critical values may be simulated by computers with arbitrary precision, thereby avoiding the problem of deriving the complicated null distribution, so that completely feasible

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procedures are available. The power properties of the four tests are investigated through Monte Carlo simulations, and, for the purpose of comparison, the frequently used test proposed by Im, Pesaran, Shin (2003) is included. The tests are applied to panels of 5, 10, 15 and 20 units with samples ranging from 30 to 500 observations under different null [OK?] and alternative parameter settings.

The paper is arranged as follows. In the next section we specify the model under investigation with the corresponding null and alternative hypotheses along with the proposed tests. The Monte Carlo simulations are presented in Section 3. In Section 4 we present the results as regards the size and power of the tests and finally, a conclusive summary is presented in Section 5.

2. Panel unit root testing

In this section we present the model and outline of the hypothesis tests. Since nearly all previously proposed successful unit root tests involved a first-order autoregressive model, with possible extensions thereof, we will follow that principle here (for a survey of approaches, see Pesaran 2007). Consider a multivariate, possibly non-stationary process defined by

$$\mathbf{Y}_t = \boldsymbol{\rho}\mathbf{Y}_{t-1} + \mathbf{u}_t \tag{2.1}$$

where $t = 1,...,T$, $\mathbf{u}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma})$ and $\boldsymbol{\Sigma} : N \times N$ is a positive semi-definite (psd) matrix of residual covariances, and $\boldsymbol{\rho} : N \times N$ is a parameter matrix where $\rho_{ij} = 0$ for $i \neq j$ by assumption. The null and alternative hypothesis of (multivariate) unit root may then be expressed by

$$H_0 : \rho_{11} = ... = \rho_{NN} = 1$$
$$H_A : \rho_{ii} < 1, \ \exists_i$$

.

In order to test this hypothesis it is convenient to take the first difference of the observable variable to get

$$\nabla \mathbf{Y}_t = \mathbf{Y}_t - \mathbf{Y}_{t-1} = (\boldsymbol{\rho} - \mathbf{I})\mathbf{Y}_{t-1} + \mathbf{u}_t. \tag{2.2}$$

On the difference form, the hypothesis may be expressed as

$$H_0 : (\rho_{11} - 1) = ... = (\rho_{NN} - 1) = 0$$
$$H_A : (\rho_{ii} - 1) < 0, \ \exists_i$$

or equivalently,

$$H_0 : (\boldsymbol{\rho} - \mathbf{I}) = \mathbf{0}$$
$$H_A : (\boldsymbol{\rho} - \mathbf{I}) \neq \mathbf{0} \quad . \tag{2.3}$$

Note that, in view of (2.2), this is simply a hypothesis of N restrictions in a linear model, which in turn is a well-established problem in multivariate analysis where test statistics are usually computed through functions of the restricted and unrestricted residual sum-of-square matrixes. A common procedure may be described as follows. Let $\mathbf{S}_R$ and $\mathbf{S}_U$ denote the restricted and unrestricted residual covariance matrixes of (2.2) and define $\mathbf{H} = T(\mathbf{S}_R - \mathbf{S}_U)$ and $\mathbf{E} = T\mathbf{S}_U$ where T is the total number of available observations after the differentiation where, by assumption, $T > N + 2$. These statistics are known as the *hypothesis* and *error* matrixes in the literature, and are the main ingredients within several well-known test statistics. Some options are presented below:

$$T_0^2 = T \cdot trace\left(\mathbf{H}\mathbf{E}^{-1}\right) \tag{2.4}$$

$$V = T \cdot trace\left(\mathbf{H}\left(\mathbf{H} + \mathbf{E}\right)^{-1}\right) \tag{2.5}$$

$$\Lambda = \det\left(\mathbf{E}\right) \big/ \det\left(\mathbf{H} + \mathbf{E}\right) \tag{2.6}$$

$$R \propto \Lambda^{-1/s}, \; s = \left(\left(N^4 - 4\right) \big/ \left(2N^2 - 5\right)\right)^{1/2}. \tag{2.7}$$

The T_0^2 statistic was initially proposed by Lawley (1938) and Hotelling (1947), and is also sometimes labeled as the 'Wald statistic'. For a one-dimensional variable this statistic is proportional to the standard Dickey-Fuller t statistic. The V statistic is known as the 'Pillai´s trace' (Pillai, 1955, 1956) and has also been derived through a Lagrangean multiplier prespective (e.g. Judge et. al. 1985). The Λ statistic is known as 'Wilks lambda', while the R statistic was introduced by Rao (1973). Further references concerning their origin, inequality ordering and null- and alternative distributions may be found in Muirhead (2005), Rao (1973), Judge et. al. (1985), Anderson (2003), Bewley (1986), Fujikoshi (1970, 1988), Kibria and Saleh (2005) and Lee (1971). These four statistics were originally derived for the purpose of testing linear hypothesis in contexts of multivariate linear models such as MANOVA and other multivariate settings. But they have also proven to be useful for testing distributional assumptions in multivariate econometric models. For example, Edgerton and Shukur (1999) and Holgersson (2004) used them to assess autocorrelation in multivariate regression and seemingly unrestricted regression (SUR) models, while Holgersson and Shukur (2004) used

them to test for multivariate heteroscedastricity. In these cases, the authors used tabulated or asymptotical critical values obtained within the references above. For non-stationary processes these critical values are not valid. But each of the four statistics is invariant to linear transformations of the kind $\mathbf{Y} \mapsto \mathbf{BY}$ for any non-singular matrix $\mathbf{B}$, and the critical values may be simulated with arbitrary precision by computers. However, unit root tests in econometrics are frequently conducted on a series of logarithms, which are not invariant to such transformations (i.e. to changes from measures using one currency to another) since the logarithm would add a constant to the series. It is therefore common to add a constant to the model to also make the statistic invariant to these changes. For example, we may write $\mathbf{Y}_t = \mathbf{a} + \mathbf{Z}_t$ where $\mathbf{Z}_t = \boldsymbol{\rho} \mathbf{Z}_{t-1} + \mathbf{u}_t$. By eliminating $\mathbf{Z}_t$ from this model we get

$$\mathbf{Y}_t = \mathbf{a}^* + \boldsymbol{\rho} \mathbf{Y}_{t-1} + \mathbf{u}_t, \quad \mathbf{a}^* = \left(\left(1 - \rho_{11}\right) a_1, ..., \left(1 - \rho_{NN}\right) a_N \right)'. \tag{2.8}$$

Note that the i:th element of $\mathbf{a}^*$ is zero when ρ_{ii} is zero. In other words, the joint hypothesis of $\left(\rho_{ii} = 1 \right) \cap \left(a_i = 0 \right)$ rules out the possibility that $\mathbf{Y}_t$ is a random walk with drift. Any of the statistics (2.4) to (2.7) may be used for testing the extended hypothesis after appropriate degrees of freedom adjustments. It is also possible to construct other tests based on the eigenvalues of $\{\lambda_n\}$ of $\mathbf{HE}^{-1}$. Perlman and Olkin (1980) showed that any test with an acceptance region $g\left(\lambda_1, ..., \lambda_N\right) \leq c$ where g is non-decreasing in each argument, is unbiased. These authors also supplied monotonicity results for the power functions of such tests. In this paper, however, we will restrict ourselves to those of (2.4) to (2.7) due to their simple functional form. In order to obtain these statistics the restricted and unrestricted sum-of-square residual matrixes need to be estimated. The most efficient way to do this is through the SUR version of model (2.2). This is defined as follows:

$$\begin{bmatrix} \nabla \mathbf{Y}_{1,t} \\ \vdots \\ \nabla \mathbf{Y}_{N,t} \end{bmatrix}_{NT \times 1} = \begin{bmatrix} \left(\mathbf{1}, \mathbf{Y}_{1,t-1}\right) & & (\mathbf{0}) \\ & \ddots & \\ (\mathbf{0}) & & \left(\mathbf{1}, \mathbf{Y}_{N,t-1}\right) \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1 \\ \vdots \\ \boldsymbol{\beta}_N \end{bmatrix} + \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_N \end{bmatrix} \text{ where } \boldsymbol{\beta}_n = \left(\alpha_n^*, \left(\rho_n - 1\right) \right)', \ n = 1, ..., N,$$

or more compactly, $\qquad \nabla \tilde{\mathbf{Y}}_t = \tilde{\mathbf{Y}}_{t-1} \boldsymbol{\beta} + \mathbf{u}_t.$ \hfill (2.9)

The tests are then conducted as follows.

i. Apply the ordinary least squares (OLS) to (2.2) and obtain the OLS residuals, $\tilde{u}_{tn}$ say.

ii. Calculate $\hat{\boldsymbol{\sigma}}_{N \times N} := \sum_{t=1}^{T} \tilde{\mathbf{u}}_t \tilde{\mathbf{u}}_t' / T$ where $\tilde{\mathbf{u}}_t = [\tilde{u}_{t1} ... \tilde{u}_{tN}]'$ and define $\mathbf{P} := (\hat{\boldsymbol{\sigma}}^{-1/2} \otimes \mathbf{I}_T)$.

iii. Transform (2.9) by the pre multiplication: $\nabla \tilde{\mathbf{Y}}_t = \tilde{\mathbf{Y}}_{t-1} \boldsymbol{\beta} + \mathbf{u}_t \mapsto$ $\mathbf{P} \nabla \tilde{\mathbf{Y}}_t = \mathbf{P} \tilde{\mathbf{Y}}_{t-1} \boldsymbol{\beta} + \mathbf{P} \mathbf{u}_t$.

iv. Apply OLS to the transformed model and calculate the unrestricted residual sum-of-square matrix $\mathbf{S}_U$ from the model $\mathbf{P} \nabla \mathbf{Y}_t = \mathbf{P} \mathbf{Y}_{t-1} \boldsymbol{\beta} + \mathbf{P} \mathbf{u}_t$ and calculate the restricted residual sum-of-square matrix $\mathbf{S}_R$ from the null model $\mathbf{P} \nabla \mathbf{Y}_t = \mathbf{P} \mathbf{u}_t$.

The $\mathbf{H}$ and $\mathbf{E}$ matrixes and the test statistics are then calculated as described above. In addition, note that this method readily extends to a model including which includes a constant term. When it comes to obtaining critical values at desired test level, these may be tabulated through simulations. However, that method is somewhat inconvenient, especially in a panel data context, since the tables would have to be constructed at two indexes (i.e. the number of cross sections and time observations respectively). An equivalent but more convenient method may be conducted through parametric bootstrap technique, described as follows.

Let $D(\mathbf{X}_0)$ be a test statistic, such as T_0^2, V, Λ or R, calculated from an original sample $\mathbf{X}_0 : T \times N$ (here $\mathbf{X}_0$ will represent $\nabla \mathbf{Y}_t$) and suppose that B independent realizations of $\mathbf{X}_0$ (say $\{\mathbf{X}_b\}_{b=1}^{B}$) are available with distributions determined by H_0 and (2.1) with the corresponding test statistics $\{D(\mathbf{X}_b)\}_{b=1}^{B}$. The empirical p-values are then determined by

$$\hat{p}_B(D(\mathbf{X}_0)) = \frac{B \hat{D}_B(\mathbf{X}_0) + 1}{B + 1} \text{ , where} \tag{2.10}$$

$$\hat{D}_B(\mathbf{X}_0) = B^{-1} \sum_{b=1}^{B} 1_{[0, \infty)}(D(\mathbf{X}_b) - D(\mathbf{X}_0)), \quad 1_A(\omega) = \begin{cases} 1, \text{ if } \omega \in A \\ 0, \text{ if } \omega \notin A \end{cases}.$$

Hence $\hat{D}_B(\mathbf{X}_0)$ is the proportion of times that $D_B(\mathbf{X}_1),...,D_B(\mathbf{X}_B)$ exceeds $D(\mathbf{X}_0)$. The critical region of the test at significance level α is:

$$\hat{p}_B(D(\mathbf{X}_0)) \leq \alpha. \tag{2.11}$$

The null hypothesis of panel unit roots is rejected on the α level when the empirical p-value is less than α. Hence, once this fairly simple algorithm is defined in appropriate software, there is no need for tabulated critical values for each $\{N, T\}$ pairing.

To sum up the discussion so far, any of these four statistics may be used to test the hypothesis of the panel data unit root of (2.8) in a sense way that simultaneously allows cross-equation correlations and heterogeneous unit roots among the different units. Critical values and/or p-values may be obtained by tabulation of critical values or, preferably, by (2.11) to obtain empirical p-values, so that the tests are indeed feasible. But there is also an issue of the power properties of the tests, in particular when they are compared with other previously proposed tests for multivariate unit roots. The most commonly used test is probably that suggested by Im, Pesaran and Shin (2003) whose test statistic is formed by averaging individual (univariate) Dickey-Fuller t-test statistics. Their test can be described as follows. Let the ith marginal model be determined by

$$\Delta y_{i,t} = a_i + \rho_i y_{i,t-1} + u_{i,t},$$

with the hypothesis

$$H_0 : \alpha_i = 0 \; \forall_i$$
$$H_A : \alpha_i < 0 \; \exists_i$$

The test statistic proposed by Im, Pesaran and Shin (2003) (IPS test) is then $\overline{t}_{NT} = N^{-1/2} \sum_{i=1}^{N} t_{iT}(p_i, \rho_i)$ where $t_{iT}(p_i, \rho_i)$ is the ith individual (marginal) Dickey Fuller t-test statistic for testing $\rho_i = 0$. Under the restriction of cross-sectional independence, the authors also derive the asymptotical distribution of the test with $Z = \sqrt{N}(\overline{t}_{NT} - E[t_{iT}])(Vart_{iT})^{-1/2} \xrightarrow{\ell} N(0,1)$ as $T, N \to \infty$, and $E[t_{iT}]$ and $\sqrt{Vart_{iT}}$ are tabulated by Monte Carlo experiments.

Where there is cross-sectional dependency, an alternative to the IPS test is the CIPS test proposed by Pesaran (2007). However, it should be noted that the CIPS test was proposed under the assumption that the cross-sectional dependency occurs due to a common factor affecting all of the variables in the panel, which may be a limitation of the test in cases where this assumption is not valid. But since these two tests are frequently used in applied analyses we will include them in the study.

The above tests, including the IPS/CIPS tests, could also be extended to allow for serially correlated errors, but here we will restrict ourselves to model (2.8) since there are already a substantial number of parameters in the model.

3. The Monte Carlo simulations

In this section we will present the data-generating process and outcomes of the experiments. The data will be generated according to the following model.

$$\mathbf{Y}_t = \mathbf{a}^* + \boldsymbol{\rho}\mathbf{Y}_{t-1} + \boldsymbol{\Sigma}^{1/2}\mathbf{u}_t, \qquad t = 1,\dots T \tag{3.1}$$

where $\mathbf{u}_t : N \times 1$ is an error term. In order to obtain the appropriate properties of the process, it has been simulated with $T+200$ observations and then the first 200 start-up values have been deleted. $\mathbf{Y}_t : N \times 1$ is then a vector, consisting of variables that are integrated of order 1, i.e. I(1), if the coefficients ρ_{ii} equals one and $\mathbf{a}^* = \left((1-\rho_{11})a_1,\dots,(1-\rho_{NN})a_N\right)'$. If ρ_{ii} is less than one the corresponding marginal variable is stationary. Our main interest lies in settings where the number of variables is rather large relative to the number of observations. We therefore use models consisting of $N \in \{5,\ 10,\ 15,\ 20\}$ and $T \in \{30,\ 50,\ 100,\ 200,\ 500\}$. These combinations of N and T include cases when the number of variables is comparable to the number of observations as well as the large sample cases (i.e. when $T \gg N$). We set $\rho_{11} = \dots = \rho_{NN} = 1$ when the size is calculated, while for the power calculations the vector $(\rho_{11},\dots,\rho_{NN})$ will be less than unity for some or all marginal variables. This will be done by using values relatively close to the null values (0.95 and 0.9 respectively). Moreover, since we are testing unit roots from a systemwize perspective, we will consider power simulations where all N marginal variables are stationary as well as cases when only a portion of the variables (π, say) is stationary. Specifically, we will consider $\pi \in \{1,\ 0.6,\ 0.2\}$. In addition, in order to impose cross dependencies we will generate the error term through the relation $\mathbf{u}_t = \mathbf{L}\boldsymbol{\eta}$ where $\boldsymbol{\eta} \square iidN(\mathbf{0},\mathbf{I})$, $\mathbf{L}\mathbf{L}' = \boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}$ is a circulant[*] matrix. This format allows for determining covariance matrixes in a high-dimensional space without having to set every σ_{ij} individually. (Note that none of the proposed test statistics (2.4) – (2.7) depends on this specific functional form of the covariance matrix. This form is merely used to conveniently generate a general, non-scalar psd covariance matrix.)

The following parameter settings generate positive definite covariance matrixes.

$\Sigma_{5\times5} = circ\,(1, 0.7, 0.5, 0.3, 0.1)$

$\Sigma_{10\times10} = circ\,(1, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, 0.01)$

$\Sigma_{10\times10} = circ\,(1, 0.7, 0.65, 0.6, 0.55, 0.5, 0.45, 0.4, 0.35, 0.3, 0.25, 0.2, 0.15, 0.1, 0.05)$

$\Sigma_{20\times20} = circ\,(1, 0.7, 0.65, 0.6, 0.55, 0.5, 0.45, 0.4, 0.35, 0.3, 0.3, 0.25, 0.25, 0.2, 0.2, 0.15, 0.15, 0.1, 0.1, 0.05, 0.05)$

*The circulant class matrixes used here have the form of $circ\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_2 \\ a_3 & a_2 & a_1 \end{pmatrix}$.

In order to evaluate the size of the tests, the following confidence interval is calculated.

$$\alpha_0 \pm 2 \frac{\sqrt{\alpha_0(1-\alpha_0)}}{M}, \tag{3.2}$$

where M is the number of Monte Carlo replications (equal to 10, 000) and α_0 is the nominal size which is equal to 0.05. The number of replicates used to determine the critical region (2.1) is set to 30,000.

4. Results

In our Monte Carlo study we calculate the estimated size by simply observing how many times the null is rejected in repeated samples under conditions where the null is true. By varying factors like those described in the previous section, we can obtain a succession of estimated sizes under different conditions. In general, the closer an estimated size is to the nominal size the better we consider a test to be. In this section the results from the Monte Carlo experiment are presented. The estimated sizes of the tests are presented in Table 1. The confidence interval in equation (3.2) is doubled in magnitude in order to emphasize the pattern of well-performing tests more clearly. Therefore, if the actual size of a test exhibits a rejection frequency significantly below 0.06 it is considered as fair, which is marked out as shaded cells in the tables (here we do not consider low type-I error as a weakness of the test).

4.1 Analysis of the Estimated Size

The sizes of the different unit root tests are presented in Table 1. The statistical tests have fair size properties in the absence of cross-sectional dependency, which is expected since it is under this circumstance that the new critical values are generated. When adding some cross-sectional dependency, the estimated type-I errors of the IPS and CIPS tests increase extremely. The results from our study indicate that the CIPS test may not account for a non-homogeneous cross-sectional dependency. The actual size does not converge toward the nominal size either, so even in large samples one should be careful when applying the IPS and

CIPS tests. Hence, pre-tests of correlation should always be used before applying any of these tests. The Lawley-Hotelling trace, Pillai´s trace, Wilk´s lambda and Rao´s multivariate F-test (T_0, Λ, V and R) are much more robust, and they are almost always unbiased. In particular, when looking at the size of the tests, the V test has shown to be very robust in all studied cases and hence should be the preferred option.

4.2 Analysis of the Estimated Power

The analysis of the power of the test is of central importance, since a test will be of little use if it does not have enough power to reject a false null hypothesis. Even if a correctly given size is a necessary prerequisite to ensure the good performance of a test, the tests should have enough power to reject a false null hypothesis. In our tables, however, we presented the power functions for all the tests though we put cross on the power results for the test with sever overrejection of the size. In this section we discuss the most interesting results of our Monte Carlo experiment; those concerning the power of the various versions of the tests. The power functions were estimated by calculating the rejection frequencies in 10,000 replications when 100%, 60% and 20% of the time series are stationary, according to the settings described in the previous section. The results of the power calculations are presented in Tables 2 to 4. In Tables 2a and 2b, we present the results of the power functions in the cases when 100% of the time series are stationary for ρ=0.90 and 0.95, respectively. For the cases when 60% and 20% of the time series are stationary, the results are presented in Tables 3a and 3b, and 4a and 4b respectively.

The tables show that the power functions satisfy the expected properties of increasing with the sample size and the percentage of the stationary time series in the data. The power decreases, however, when the number of equations, and the strength of dependency among the equations, increases. The power functions are very low in small samples and large numbers of equations. In cases when 20% of the time series are stationary and small samples there is very little difference between the estimated size and the estimated power.

For the IPS and CIPS tests, introducing cross-sectional dependency also has an impact on the power properties of the tests. However, due to the fact that these tests are not robust in this situation and have improper sizes, the results of the power should not be considered any more

in the analysis. In situations where there is cross-sectional dependency, the LM test should be preferred since, in general, it has the highest power among the tests for unit roots.

Table 1: Sizes of the tests

T	Uncorrelated error terms						Cross-correlated error terms					
	IPS	CIPS	T_0	Λ	V	R	IPS	CIPS	T_0	Λ	V	R
						N=5						
30	0.048	0.048	0.051	0.050	0.048	0.050	0.106	0.194	0.059	0.059	0.057	0.059
50	0.050	0.049	0.049	0.049	0.049	0.049	0.106	0.215	0.053	0.054	0.055	0.054
100	0.051	0.052	0.049	0.048	0.050	0.048	0.114	0.232	0.055	0.057	0.057	0.057
200	0.048	0.047	0.052	0.052	0.051	0.052	0.109	0.230	0.059	0.059	0.058	0.059
500	0.051	0.050	0.053	0.052	0.052	0.052	0.110	0.230	0.060	0.059	0.059	0.059
T						N=10						
30	0.051	0.048	0.061	0.062	0.058	0.062	0.144	0.242	0.051	0.047	0.051	0.047
50	0.045	0.049	0.045	0.045	0.052	0.045	0.148	0.264	0.059	0.057	0.060	0.057
100	0.048	0.056	0.049	0.049	0.047	0.049	0.145	0.261	0.061	0.062	0.063	0.062
200	0.055	0.049	0.058	0.058	0.055	0.058	0.153	0.280	0.060	0.060	0.060	0.060
500	0.055	0.055	0.055	0.056	0.056	0.056	0.158	0.284	0.069	0.069	0.069	0.069
T						N=15						
30	0.050	0.049	0.048	0.045	0.048	0.045	0.196	0.312	0.061	0.037	0.052	0.037
50	0.052	0.053	0.051	0.050	0.051	0.050	0.202	0.318	0.055	0.048	0.054	0.048
100	0.048	0.052	0.055	0.054	0.053	0.005	0.199	0.338	0.059	0.060	0.062	0.007
200	0.046	0.047	0.043	0.043	0.045	0.043	0.193	0.338	0.057	0.058	0.059	0.058
500	0.051	0.054	0.050	0.050	0.050	0.050	0.204	0.352	0.068	0.069	0.070	0.069
T						N=20						
30	0.051	0.053	0.049	0.050	0.046	0.050	0.208	0.299	0.085	0.029	0.046	0.029
50	0.047	0.047	0.047	0.047	0.048	0.047	0.207	0.317	0.053	0.037	0.047	0.037
100	0.049	0.050	0.050	0.050	0.049	0.050	0.212	0.335	0.050	0.048	0.053	0.048
200	0.052	0.047	0.050	0.050	0.052	0.050	0.203	0.330	0.064	0.064	0.060	0.064
500	0.051	0.050	0.050	0.050	0.050	0.050	0.208	0.335	0.064	0.066	0.067	0.066

Table 2a: Power when 100% of the time series are stationary where ρ=0.90

T	Uncorrelated error terms						Cross-correlated error terms					
	IPS	CIPS	T_0	Λ	V	R	IPS	CIPS	T_0	Λ	V	R
						N=5						
30	0.198	0.209	0.124	0.151	0.172	0.151	x	x	0.139	0.165	0.186	0.165
50	0.471	0.477	0.319	0.354	0.390	0.354	x	x	0.333	0.369	0.406	0.369
100	0.975	0.963	0.919	0.929	0.939	0.929	x	x	0.922	0.934	0.941	0.934
200	1.000	1.000	1.000	1.000	1.000	1.000	x	x	1.000	1.000	1.000	1.000
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	1.000	1.000	1.000	1.000
T						N=10						
30	0.374	0.395	x	x	0.279	0.220	x	x	0.137	0.211	0.261	0.211
50	0.777	0.787	0.445	0.552	0.620	0.552	x	x	0.467	0.565	0.615	0.565
100	1.000	1.000	0.996	0.998	0.998	0.998	x	x	0.997	0.997	0.998	0.997
200	1.000	1.000	1.000	1.000	1.000	1.000	x	x	1.000	1.000	1.000	1.000
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	x	x
T						N=15						
30	0.504	0.530	0.105	0.227	0.275	0.227	x	x	0.125	0.201	0.238	0.201
50	0.926	0.924	0.483	0.663	0.741	0.663	x	x	0.481	0.633	0.640	0.633
100	1.000	1.000	1.000	1.000	1.000	0.994	x	x	1.000	1.000	1.000	0.993
200	1.000	1.000	1.000	1.000	1.000	1.000	x	x	1.000	1.000	1.000	1.000
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	x	x
T						N=20						
30	0.635	0.643	0.067	0.228	0.202	0.228	x	x	x	0.166	0.139	0.166
50	0.978	0.973	0.464	0.710	0.749	0.710	x	x	0.468	0.650	0.582	0.650
100	1.000	1.000	1.000	1.000	1.000	1.000	x	x	1.000	1.000	1.000	1.000
200	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	1.000	x
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	x	x

Table 2b: Power when 100% of the time series are stationary where ρ=0.95

	Uncorrelated error terms						Cross-correlated error terms					
						N=5						
T	IPS	CIPS	T_0	Λ	V	R	IPS	CIPS	T_0	Λ	V	R
30	0.097	0.098	0.073	0.083	0.087	0.083	x	x	0.084	0.094	0.104	0.094
50	0.162	0.170	0.114	0.125	0.139	0.125	x	x	0.130	0.141	0.154	0.141
100	0.457	0.462	0.352	0.368	0.384	0.368	x	x	0.363	0.380	0.397	0.380
200	0.973	0.961	0.931	0.935	0.938	0.935	x	x	0.939	0.944	0.948	0.944
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	1.000	1.000	1.000	1.000
T						N=10						
30	0.145	0.157	0.074	0.104	0.121	0.104	x	x	0.086	0.102	0.126	0.102
50	0.272	0.296	0.152	0.183	0.211	0.183	x	x	0.162	0.198	0.234	0.198
100	0.773	0.785	0.560	0.603	0.637	0.603	x	x	0.585	0.628	0.663	0.628
200	0.820	0.820	0.817	0.817	0.818	0.817	x	x	0.998	0.999	0.999	0.999
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	x	x
T						N=15						
30	0.185	0.195	0.069	0.110	0.131	0.110	x	x	0.084	0.093	0.117	0.093
50	0.378	0.407	0.164	0.227	0.270	0.227	x	x	0.169	0.220	0.254	0.220
100	0.921	0.923	0.690	0.749	0.791	0.350	x	x	0.700	0.753	0.784	0.358
200	1.000	1.000	1.000	1.000	1.000	1.000	x	x	1.000	1.000	1.000	1.000
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	x	x
T						N=20						
30	0.228	0.235	0.055	0.106	0.106	0.106	x	x	x	0.072	0.088	0.072
50	0.497	0.503	0.160	0.265	0.299	0.265	x	x	0.168	0.213	0.227	0.213
100	0.975	0.974	0.748	0.831	0.874	0.831	x	x	0.773	0.838	0.852	0.838
200	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	1.000	x
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	x	x

Table 3a: Power when 60% of the time series are stationary where ρ=0.90

	Uncorrelated error terms						Cross-correlated error terms					
						N=5						
T	IPS	CIPS	T_0	Λ	V	R	IPS	CIPS	T_0	Λ	V	R
30	0.120	0.124	0.092	0.103	0.108	0.103	x	x	0.103	0.108	0.113	0.108
50	0.226	0.226	0.170	0.184	0.197	0.184	x	x	0.228	0.234	0.240	0.234
100	0.577	0.599	0.568	0.572	0.580	0.572	x	x	0.745	0.743	0.741	0.743
200	0.968	0.980	0.995	0.996	0.995	0.996	x	x	1.000	1.000	1.000	1.000
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	1.000	1.000	1.000	1.000
T						N=10						
30	0.187	0.196	0.091	0.129	0.148	0.129	x	x	0.112	0.131	0.143	0.131
50	0.360	0.381	0.228	0.272	0.301	0.272	x	x	0.282	0.314	0.329	0.314
100	0.858	0.861	0.820	0.837	0.846	0.837	x	x	0.908	0.913	0.912	0.913
200	1.000	1.000	1.000	1.000	1.000	1.000	x	x	1.000	1.000	1.000	1.000
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	x	x
T						N=15						
30	0.228	0.245	0.074	0.121	0.139	0.121	x	x	0.112	0.117	0.119	0.117
50	0.503	0.523	0.252	0.340	0.367	0.340	x	x	0.312	0.350	0.327	0.350
100	0.958	0.965	0.923	0.938	0.940	0.664	x	x	0.964	0.969	0.959	0.785
200	1.000	1.000	1.000	1.000	1.000	1.000	x	x	1.000	1.000	1.000	1.000
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	x	x
T						N=20						
30	0.295	0.301	0.067	0.133	0.113	0.133	x	x	x	0.094	0.070	0.094
50	0.621	0.618	0.247	0.371	0.378	0.371	x	x	0.305	0.339	0.276	0.339
100	0.991	0.990	0.955	0.968	0.969	0.968	x	x	0.981	0.982	0.967	0.982
200	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	1.000	x
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	x	x

Table 3b: Power when 60% of the time series are stationary where ρ=0.95

	Uncorrelated error terms						Cross-correlated error terms					
						N=5						
T	IPS	CIPS	T_0	Λ	V	R	IPS	CIPS	T_0	Λ	V	R
30	0.077	0.077	0.067	0.071	0.074	0.071	x	x	0.068	0.073	0.078	0.073
50	0.097	0.102	0.079	0.082	0.086	0.082	x	x	0.104	0.107	0.109	0.107
100	0.215	0.219	0.183	0.187	0.195	0.187	x	x	0.104	0.107	0.109	0.107
200	0.568	0.576	0.569	0.574	0.578	0.574	x	x	0.766	0.766	0.763	0.766
500	0.993	0.998	1.000	1.000	1.000	1.000	x	x	1.000	1.000	1.000	1.000
T						N=10						
30	0.092	0.100	0.063	0.079	0.087	0.079	x	x	0.072	0.077	0.085	0.077
50	0.151	0.159	0.105	0.120	0.131	0.120	x	x	0.118	0.132	0.144	0.132
100	0.352	0.370	0.270	0.291	0.308	0.291	x	x	0.344	0.356	0.367	0.356
200	0.844	0.848	0.837	0.841	0.846	0.841	x	x	0.919	0.920	0.921	0.920
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	x	x
T						N=15						
30	0.117	0.125	0.061	0.085	0.091	0.085	x	x	0.074	0.060	0.074	0.060
50	0.191	0.202	0.109	0.136	0.154	0.136	x	x	0.116	0.129	0.138	0.129
100	0.498	0.512	0.359	0.392	0.413	0.105	x	x	0.434	0.457	0.462	0.137
200	0.949	0.952	0.943	0.948	0.951	0.948	x	x	0.978	0.978	0.978	0.978
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	x	x
T						N=20						
30	0.136	0.133	0.056	0.081	0.080	0.081	x	x	x	0.053	0.057	0.053
50	0.234	0.240	0.104	0.141	0.155	0.141	x	x	0.120	0.113	0.114	0.113
100	0.621	0.613	0.401	0.459	0.493	0.459	x	x	0.466	0.499	0.496	0.499
200	0.987	0.985	0.983	0.984	0.986	0.984	x	x	x	x	0.994	x
500	1.000	1.000	1.000	1.000	1.000	1.000	x	x	x	x	x	x

Table 4a: Power when 20% of the time series are stationary where ρ=0.90

	Uncorrelated error terms						Cross-correlated error terms					
						N=5						
T	IPS	CIPS	T_0	Λ	V	R	IPS	CIPS	Wald	LR	LM	Rao
30	0.065	0.062	0.060	0.061	0.063	0.061	x	x	0.071	0.070	0.069	0.070
50	0.085	0.087	0.079	0.080	0.084	0.080	x	x	0.112	0.106	0.103	0.106
100	0.134	0.141	0.146	0.145	0.146	0.145	x	x	0.314	0.296	0.280	0.296
200	0.269	0.278	0.390	0.382	0.371	0.382	x	x	0.814	0.797	0.777	0.797
500	0.652	0.681	0.980	0.977	0.974	0.977	x	x	1.000	1.000	1.000	1.000
T						N=10						
30	0.084	0.087	0.057	0.069	0.077	0.069	x	x	0.072	0.068	0.070	0.068
50	0.106	0.110	0.092	0.099	0.105	0.099	x	x	0.126	0.126	0.122	0.126
100	0.142	0.140	0.148	0.151	0.150	0.151	x	x	0.374	0.360	0.340	0.360
200	0.438	0.433	0.631	0.615	0.596	0.615	x	-	0.901	0.885	0.866	0.885
500	0.955	0.964	1.000	1.000	1.000	1.000	x	x	x	x	x	x
T						N=15						
30	0.092	0.094	0.064	0.071	0.073	0.071	x	x	0.082	0.057	0.057	0.057
50	0.133	0.133	0.092	0.108	0.115	0.108	x	x	0.127	0.119	0.110	0.119
100	0.261	0.263	0.271	0.277	0.276	0.062	x	x	0.452	0.425	0.390	0.131
200	0.564	0.561	0.772	0.759	0.741	0.759	x	x	0.964	0.953	0.936	0.953
500	0.974	0.974	1.000	1.000	1.000	1.000	x	x	x	x	x	x
T						N=20						
30	0.104	0.101	0.056	0.070	0.063	0.070	x	x	x	0.048	0.043	0.048
50	0.152	0.150	0.095	0.115	0.108	0.115	x	x	0.130	0.106	0.088	0.106
100	0.317	0.312	0.308	0.321	0.310	0.321	x	x	0.465	0.440	0.382	0.440
200	0.684	0.645	0.874	0.860	0.837	0.860	x	x	x	x	0.959	x
500	0.994	0.992	1.000	1.000	1.000	1.000	x	x	x	x	x	x

Table 4b: Power when 20% of the time series are stationary where ρ=0.95

T	Uncorrelated error terms						Cross-correlated error terms					
	IPS	CIPS	T_0	Λ	V	R	IPS	CIPS	T_0	Λ	V	R
						N=5						
30	0.058	0.057	0.060	0.060	0.058	0.060	x	x	0.063	0.061	0.062	0.061
50	0.065	0.064	0.058	0.059	0.060	0.059	x	x	0.071	0.069	0.070	0.069
100	0.082	0.082	0.076	0.078	0.079	0.078	x	x	0.117	0.113	0.112	0.113
200	0.140	0.144	0.151	0.152	0.150	0.152	x	x	0.317	0.310	0.299	0.310
500	0.334	0.346	0.524	0.518	0.513	0.518	x	x	0.924	0.921	0.915	0.921
T						N=10						
30	0.068	0.070	0.053	0.058	0.064	0.058	x	x	0.063	0.057	0.060	0.057
50	0.072	0.070	0.064	0.066	0.072	0.066	x	x	0.075	0.072	0.076	0.072
100	0.105	0.108	0.097	0.099	0.100	0.099	x	x	0.137	0.136	0.132	0.136
200	0.199	0.194	0.213	0.212	0.212	0.212	x	x	0.364	0.355	0.347	0.355
500	0.708	0.728	0.908	0.913	0.920	0.911	x	x	x	x	x	x
T						N=15						
30	0.063	0.063	0.051	0.057	0.058	0.057	x	x	0.068	0.043	0.056	0.043
50	0.081	0.084	0.060	0.069	0.073	0.069	x	x	0.070	0.064	0.070	0.064
100	0.126	0.132	0.114	0.120	0.123	0.018	x	x	0.156	0.156	0.152	0.026
200	0.249	0.251	0.264	0.265	0.266	0.265	x	x	0.458	0.446	0.434	0.446
500	0.695	0.683	0.935	0.930	0.925	0.930	x	x	x	x	x	x
T						N=20						
30	0.069	0.069	0.052	0.058	0.059	0.058	x	x	x	0.032	0.044	0.032
50	0.089	0.088	0.061	0.068	0.074	0.068	x	x	0.075	0.057	0.061	0.057
100	0.145	0.153	0.113	0.122	0.128	0.122	x	x	0.153	0.154	0.148	0.154
200	0.314	0.292	0.335	0.334	0.337	0.334	x	x	x	x	0.491	x
500	0.792	0.767	0.975	0.972	0.969	0.972	x	x	x	x	x	x

5. Summary and Conclusions

In this paper we propose four tests for unit roots from panel data perspectives. These tests are generalized from traditional multivariate tests for linear restrictions and have the properties of being valid even under unknown cross-sectional dependence, a property that is present in many real applications. In fact, the test statistics known as Hotellings trace, Lawley-Hotellings trace, Wilks lambda and Rao´s multivariate F-test were developed in the early and mid-nineteen hundreds to deal with precisely these matters. The tests do not, in contrast to previously introduced panel data unit root tests, assume a common (homogeneous) unit root parameter in each marginal model or cross-sectional independence. Moreover, the fact that the test statistics are invariant under certain linear transformations, enables the use of Monte Carlo critical values so that there is no need to derive null distributions of the test statistics. Arbitrary size precision is therefore available through computer simulations. In these ways, the proposed tests are universally valid and rely on a minimum of distributional assumptions. However, before a new statistical method can be considered useful, its properties should be examined and contrasted to existing, previously proposed methods. In this perspective, we

have conducted a Monte Carlo simulation of the four tests as well as the common tests labeled as IPS and CIPS respectively. A large number of models were investigated in the simulations, where the number of observations, the number of equations and the degree of dependency between equations were varied. For the power calculations, the deviation from the null hypothesis was varied through different percentages of stationary marginal variables and different sizes of the autoregressive parameter. The results have shown that IPS and CIPS are non-robust to cross-sectional dependency. These tests considerably over-reject the null hypothesis when it is true. The tests of Hotellings trace, Lawley-Hotellings trace, Wilks lambda and Rao´s multivariate F-test are demonstrated to have stable size properties in the sense that they remain close to the nominal size even under heterogeneous alternatives and the presence of cross-correlations.

The power functions have been shown to increase with the sample size and the percentage of the stationary time series in the data, but decrease when the number of equations and strength of dependency among the equations increases, as expected. Since the IPS and CIPS tests do not have fair size properties, their power properties cannot be judged in an appropriate way under a general covariance matrix. However, in cases where the number of cross units N is proportional to, or even exceeds, the number of observations T, some sorts of parameter restriction, such as those of the CIPS or IPS test, need to be imposed in order to obtain a consistent test. In other words, hypothesis testing of multivariate unit roots is concerned with a trade-off between on the one hand requiring high power in cases where N and T are proportional (which requires strong restrictions on the functional form of the covariance matrix and/or the unit root parameter matrix) and on the other hand requiring a test which remains well behaved under a general residual covariance matrix of unknown functional form. The conclusions of our paper are that in cases where the data-generating process is *a priori* known to follow a simple structure, such as a factor model or a diagonal covariance matrix, tests developed under such conditions will be the optimal choice. However, in more general contexts usage of these tests may be hazardous and should be abandoned in favor of multivariate tests. In particular, this paper shows that the Lawley-Hotelling trace test has the greatest size and power properties among other comparable tests and is hence the test to be recommended in real applications.

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