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Abstract

In this paper, a number of procedures have been proposed for developing new biased estimators of seemingly unrelated regression (SUR) parameters, when the explanatory variables are affected by multicollinearity. Several ridge parameters are proposed and then compared in terms of the trace mean squared error (TMSE) and(PR) criterion. The PR is the proportion of replication (out of 1,000) for which the SUR version of the generalised least squares, (SGLS) estimator has a smaller TMSE than the others. The study has been made using Monte Carlo simulations where the number of equations in the system, number of observations, correlation among equations and correlation between explanatory variables have been varied. For each model we performed 1,000 replications. Our results show that under certain conditions the performance of the multivariate regression estimators based on SUR ridge parameters R_{Sarith}, $R_{Sqarith}$ and R_{Smax} are superior to other estimators in terms of TMSE and PR criterion. In large samples and when the collinearity between the explanatory variables is not high the unbiased SUR, estimator produces a smaller TMSEs.

Key words: Multicollinearity; SUR ridge regression; Monte Carlo simulations; biased estimators; Generalized least squares.

Mathematics Subject classification 62 J07.

1. Introduction

The seemingly unrelated regressions (SUR) model proposed by Zellner (1962) is considered as one of the most successful and efficient methods for estimating seemingly unrelated regressions and tests of aggregation bias. The resulting (SUR) model has simulated a countless theoretical and empirical results in econometrics and other areas, (see Zellner, 1962; Srivastava and Giles, 1987; Chib and Greenberg, 1995). For example, the methodology is applicable to allocation models, demand functions for a number of commodities, investment functions for a number of firms, income or consumption functions for subsets of populations or different regions, to mention some.

In most of the empirical works people are often concerned about problems with the specification of the model or problems with the data. In the sequel, our interest lies in data type problem, namely multicollinearity. This problem arises in situations when the explanatory variables are highly inter-correlated. Then it becomes difficult to disentangle the separate effects of each of the explanatory variables on the explained variable. As a result, the estimated parameters may be statistically insignificant and/or have (unexpectedly) different signs. Thus, to conduct meaningful statistical inference would be difficult for the researcher.

In this paper, we shall consider the use of iterative procedures to improve the efficiency of estimators which are unbiased under some mild restrictions on random errors. An important class of estimators which may be considered this way is that of ridge regression estimators. This class contains estimators which although may be biased may have smaller MSEs than their unbiased counterparts. This type of shrinkage estimator was originally developed to deal with the problem of multicollinearity in the linear regression model. History of multicollinearity dates back at least to the paper published by Ragnar A. Frisch (1934) who introduced the concept of multicollinearity to designate a situation where the variables dealt with are subject to two or more relations. Studies on ridge regression estimators are pioneered by Hoerl and Kennard (1970a, 1970b), and later followed by Vinod (1978), Brown and Zidek (1980), Srivastava and Giles (1987), Haitovsky (1987) Saleh and Kibria (1993), and Firinguetti (1997). In the single equation environment, Kibria (2003) and Alkhamisi, et. al. (2006) used simulation techniques to study the properties of some new proposed estimators and compared their properties with other popular existing estimators. Under certain conditions, they found that the MSEs of some of the new proposed estimators are smaller than the corresponding MSE of the OLS estimator and other known existing estimators.

However, we discuss here more thoroughly the problems associated with system-wise ridge estimation using different multivariate ridge parameters, since this topic is often only briefly mentioned, if at all, in the literature. In Tables 1-10, we present the TMSE and PR values for the proposed multivariate ridge parameters and further discussion with regard to their construction is to be discussed in the next section. The first parameter however is the SGLS estimate,(eq. 7), while the following six SUR ridge parameters are SUR versions of those proposed by Kibiria (2003) and / or Alkhamisi, et. al. (2006), while the last three parameters are our proposed new parameters. We also conducted a study for equivalent cases to the proposed multivariate ridge parameters but when the explanatory variables are generated from a multivariate T- distribution. The TMSE for these parameters are presented in Tables 3-6. In these tables the PR values (in percentage) are placed in parenthesis below the values of the corresponding estimators.

For further studies related to ridge regression in multiple regression and /or in SUR model we refer the interested reader to Brown and Zidek (1980), Haitovsky (1987), Srivastava and Giles (1987), Firinguetti (1997) and Firebig and Kim (2000) among others.

The paper is organised as follows: In the next section we present the model studied, and define a number of variants of ridge regression parameters applicable in SUR model. Section 3, describes the Monte Carlo experiment together with the factors that can affect the properties of the proposed parameters. In Section 4, we present the results concerning the various ridge parameters in terms of TMSE and PR criterion. The conclusions of the paper are presented in Section 5.

2. Methodology

Consider a system of M equations given by

$$Y_i = X_i B_i + e_i, \qquad i = 1, 2, \ldots, M \qquad (1)$$

where $\mathbf{Y}_i$ is a $T \times 1$ vector of observations on the dependent variable, e_i is a $T \times 1$ vector of random errors with $E(e_i)=0$, X_i is a $T \times n_i$ matrix of observations on n_i explanatory variables including a constant term, and B_i is a $n_i \times 1$ dimensional vector of unknown location parameters . M is the number of equations in the system, T is the number of observations per equation, and n_i is the number of rows in the vector B_i .

Let $Y = (Y_1', Y_2', ..., Y_M')'$, $X = \mathrm{diag}(X_1, X_2, ..., X_M)$, and similarly e and B are defined. Then the M equations in (1) can be written in the compact form

$$Y = XB + e\,, \qquad (2)$$

where Y and e are each of dimension TM×1, X is of dimension TM×n, $n = \sum_{i=1}^{M} n_i$, and B is a n-dimensional vector of location parameters.

Furthermore, we have to make the following assumptions:

a) $\mathbf{X_i}$ is fixed with rank n_i.

b) $\mathrm{plim}\,\frac{1}{T}(X_i'X_i) = Q_{ii}$ is non-singular with finite and fixed elements, i.e. invertable.

c) Assume that $\mathrm{plim}\,\frac{1}{T}(X_i'X_j) = Q_{ij}$ is non-singular with finite and fixed elements.

d) $E(e_i e_j') = \sigma_{ij} I_T$, where σ_{ij} designate the covariance between the i^{th} and j^{th} equations for each observation in the Sample. The above expression can be written as

$E(e)=0$ and $E(ee') = \Sigma \otimes I_T$, where $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & . & \sigma_{1M} \\ \sigma_{21} & \sigma_{22} & . & \sigma_{2M} \\ . & . & . & . \\ \sigma_{M1} & \sigma_{M2} & . & \sigma_{MM} \end{bmatrix}$ is an M×M positive definite symmetric matrix and $\otimes$ represent the kronecker product.

Thus the errors at each equation are assumed to be homoscedastic and not autocorrelated, but that there is contemporaneous correlation between corresponding errors in different equations.
The OLS estimator of B in (2) is

$$\hat{B} = (X'X)^{-1}X'Y\,, \quad \text{with} \qquad (3)$$
$$\mathrm{var}(\hat{B}) = (X'X)^{-1}X'(\Sigma^{-1} \otimes I_T)\,X\,(X'X)^{-1}.$$

In the context of SUR model, Srivastava and Giles(1987) defined the general ridge estimator of B as

$$\hat{B}_{OR} = (X'X + R)^{-1} X'Y \tag{4}$$

where R is an nxn matrix of nonnegative elements. But the ridge estimator in (4) does not include the cross equation correlation among errors. For this, the following transformation is suggested, (see Srivastava and Giles, 1978).

$$Y^* = (\Sigma^{-1/2} \otimes I_T)\, Y \,,\; X^* = (\Sigma^{-1/2} \otimes I_T)\, X \,, \;\text{ and }\; e^* = (\Sigma^{-1/2} \otimes I_T)\, e \,.$$

Using the above transformation the model in (2) can be expressed as

$$Y^* = X^*B + e^*, \tag{5}$$

where Y* and e* are $MT \times 1$ vectors, X* is an $MT \times n$ matrix, $E(e^*e^{*'})=I_{TM}$ and $E(e^*)=0$.

Accordingly, the GLS estimator of B in (5) and its ridge version are respectively given by

$$\hat{B}_G = (X^{*'}X^*)^{-1} X^{*'}Y^* \tag{6}$$

$$\hat{B}_{GR} = (X^{*'}X^* + R)^{-1} X^{*'}Y^*, \text{ with} \tag{7}$$

$$MSE(\hat{B}_{GR}) = (X^{*'}X^* + R)^{-1}(RBB'R' + X^{*'}X^*)(X^{*'}X^* + R)^{-1}$$

Set R = 0 in the expression for $MSE(\hat{B}_{GR})$ to obtain an expression for $MSE(\hat{B}_G)$.

Let Λ and Ψ designate the eigenvalues and eigenvectors of $(X^{*'}X^*)$ respectively. Then $\Psi'(X^{*'}X^*)\Psi = \Lambda$ and the canonical version of model (5) is given by

$$Y^* = Z\alpha + e^*, \qquad (8)$$

where $Z = X^*\Psi$, $\alpha = \Psi' B$ and $Z'Z = (\Psi' X^{*'} X^* \Psi) = \Lambda$.

The corresponding GLS estimator of α is

$$\hat{\alpha} = (Z'Z)^{-1} Z'Y^* \qquad (9)$$

and the expression for the corresponding SUR-ridge regression parameter is

$$\hat{\alpha}_{SUR} = (Z'Z + R)^{-1} Z'Y^*, \qquad (10)$$

where $R = diag(R_1, R_2, \dots, R_M)$, $R_i = diag(r_{i1}, r_{i2,\dots,} r_{in_i})$ and $r_{ij} > 0$ for i = 1, 2, …, M. Moreover, the bias vector, the mean squared error matrix of $\hat{\alpha}_{SUR}$ and trace of MSE($\hat{\alpha}_{SUR}$) (TMSE) are respectively given by

$$E(\hat{\alpha}_{SUR}) - \alpha = -(Z'Z + R)^{-1} R\alpha, \qquad (11)$$

$$\begin{aligned} MSE(\hat{\alpha}_{SUR}) &= E(\hat{\alpha}_{SUR} - \alpha)(\hat{\alpha}_{SUR} - \alpha)' \\ &= [(\Lambda + R)^{-1}\Lambda - I]\alpha\alpha'[(\Lambda + R)^{-1}\Lambda - I]' + (\Lambda + R)^{-1}\Lambda(\Lambda + R)^{-1} \\ &= (\Lambda + R)^{-1}(\Lambda + R\alpha\alpha' R')(\Lambda + R)^{-1} \end{aligned} \qquad (12)$$

and

$$TMSE(\hat{\alpha}_{SUR}(R)) = \sum_{i=1}^{M}\sum_{j=1}^{n_i} \frac{\lambda_{ij} + r_{ij}^2\alpha_{ij}^2}{(\lambda_{ij} + r_{ij})^2} \qquad (13)$$

Now set $\dfrac{\partial TMSE(\hat{\alpha}_{SUR})}{\partial r_{ij}} = 0$, to determine the optimum values of r_{ij}

$$r_{ij} = \frac{1}{\hat{\alpha}_{ij}^2} . \tag{14}$$

Moreover, conditions to ensure the superiority of $\hat{\alpha}_{SUR}(R)$ over $\hat{\alpha}$ with respect to the MSE criterion are given in the following result.

Result 1.

a. $MSE(\hat{\alpha}) - MSE(\hat{\alpha}_{SUR}(R))$ is a positive semidefinite matrix iff

$$\alpha'(\Lambda^{-1} + 2R^{-1})^{-1}\alpha \;\; \leq \;\; 1 \tag{15}$$

b. Sufficient conditions for (15) to hold are

(i) $\alpha'\Lambda\alpha \leq 1$ (ii) $\alpha' R\alpha \leq 2$. (16)

c. Set R = rI in (10), to show that MSE $(\hat{\alpha}) - MSE(\hat{\alpha}_{SUR}(R))$ is

a positive semidefinite matrix if $r \leq \dfrac{2}{\alpha'\alpha}$. (17)

The following result presents some new methods for constructing SUR-ridge parameters.

Result 2.

For j = 1, … , n_i; i = 1,…, M, assume eq. (14) holds then

1. R_{SK}. The ij-th component of this matrix is given by (14), (see Srivastava and Giles, 1987 and Firinguetti, 1997).
2. R_{SHK}. Denotes the SUR version of Hoerl and Kennard (1970a) ordinary ridge parameter.

$$r_{ij(SHK)} = \frac{1}{\max\limits_{ij}(\hat{\alpha}_{ij}^2)} \tag{18}$$

3. R_{Sharm}. Designates the SUR version to the harmonic mean proposed by Hoerl, Kennard and Baldwin (1975), (see Firinguetti, 1997).

$$r_{ij(Sham)} = \frac{n}{\sum\limits_{i=1}^{M}\sum\limits_{j=1}^{n_i}\frac{1}{r_{ij}}} = \frac{n}{\sum\limits_{i=1}^{M}\sum\limits_{j=1}^{n_i}\hat{\alpha}_{ij}^2} \tag{19}$$

4. R_{Sarith}. A SUR extension to the single equation arithmetic mean proposed by Kibria (2003).

$$r_{ij(Sarith)} = \frac{1}{n}\sum_{i=1}^{M}\sum_{j=1}^{n_i}\frac{1}{\hat{\alpha}_{ij}^2} \tag{20}$$

5. R_{Sgeom}. A generalization to the single equation geometric mean proposed by Kibria (2003).

$$r_{ij(Sgeom)} = \frac{1}{\left(\prod\limits_{i=1}^{M}\prod\limits_{j=1}^{n_i}\hat{\alpha}_{ij}^2\right)^{\frac{1}{n}}} \tag{21}$$

6. R_{Skmed}. The median of r_{ij} in (14) is used to define this parameter, (see Kibria, 2003 for a single equation version).

$$r_{ij(Skmed)} = \underset{ij}{\text{median}}\left(\frac{1}{\hat{\alpha}_{ij}^2}\right) \tag{22}$$

7. $R_{Sqarith}$ (New). We propose a ridge parameter using the arithmetic mean of $\sqrt{r_{ij}}$, with r_{ij} as defined in (14).

$$r_{ij(Sqarith)} = \underset{ij}{\text{mean}}\left(\frac{1}{\sqrt{\hat{\alpha}_{ij}^2}}\right) \tag{23}$$

8. R_{Sqmax} (New). We propose a new ridge parameter based on the maximization of $\sqrt{r_{ij}}$,

with r_{ij} as defined in (14).

$$r_{ij(Sqmax)} = \max_{ij}\left(\frac{1}{\sqrt{\hat{\alpha}_{ij}^2}}\right) \qquad (24)$$

9. R_{Smax} (New). A generalization to the single equation ridge parameter K_{max}^{HK} proposed by Alkhamisi et. al. (2006).

$$r_{ij(Smax)} = \max_{ij}\left(\frac{1}{\hat{\alpha}_{ij}^2}\right) \qquad (25)$$

Clearly all of the ridge estimators defined by eqs. (18) - (22) and eq. (25) are identical to R_{SHK} when $\hat{\alpha}_{ij}^2$ is replaced by $\max(\hat{\alpha}_{ij}^2)$. The estimators in eqs. (18) - (19) have already been considered by Firinguetti (1997).

In order to assess the performance of multivariate ridge regression estimators defined in terms of the above proposed multivariate ridge estimators we performed a Monte Carlo experiment to compare them in terms of TMSE with the GLS estimator, (see eq. 6) and the general ridge regression estimator defined by eq. (7) and eq. (14). The first two column in Tables 1-10, (SGLS and SK), present the TMSE for the GLS estimator and the general ridge estimator, respectively.

3. The Monte Carlo Experiment

In this work, a criterion to compare between the different SUR-ridge type estimators of the unknown vector parameter B is required. The criterion proposed to measure the goodness of an estimator of B, say $\tilde{B}$, are the TMSE and the PR criterion. The total mean square error is defined as

$$\text{TMSE}(\tilde{B}) = \text{Trace}[\,E(\tilde{B} - B)(\tilde{B} - B)'\,].$$

The PR criterion counts the proportion of replications,(out of 1000), for which the SUR version of generalized least square estimator(SGLS) produces a smaller TMSE than the remaining multivariate ridge estimators. In Tables 1-10 these numbers are placed in parenthesis. The performance of the different SUR ridge estimators, under consideration, are examined via Monte Carlo simulations. The Monte Carlo experiment has been performed by generating data in accordance with the following equation

$$y_{ti} = \sum_{j=1}^{5} x_{tij}\beta_{ij} + e_{ti}\ , \quad t = 1, 2, \cdots, T;\ i = 1, 2, \cdots, M, \qquad (26)$$

where $x_{ti1=1}$. The explanatory variables are generated from $MVN_4(0, \Sigma_x)$, multivariate T(2) and multivariate T(6), respectively,(see Appendix A). The random errors were generated from $MVN_m(0, \Sigma_e)$, m = 3, 10, (see Tables 1-10). For each model we have performed 1,000 replications using the statistical software S-plus version 6.0. Tables 1-10, present values of factors that varied in simulations which may or may not affect the properties of the proposed multivariate ridge parameters.

3.1 Algoritm

The simulation algorithm is based on the following steps.

a. Generate the explanatory variables from $MVN_4(0, \Sigma_x)$, T(6) or T(2).
b. Set initial value of B either to $(1,1,1,1,1)'$ or $(1, 2, 3, 4, 5)'$.
c. Simulate the vector random error e from $MVN_m(0, \Sigma_e)$, m = 3, 10.
d. As outlined earlier, for a given X structure, transform the original model (2) to an orthogonal form given by eq. (8) and calculate the SGLS estimator along with $\hat{\alpha}_{SUR}(R)$, R= R_{SK}, R_{SHK}, $R_{Sharm,}$ R_{Sarith}, R_{Sgeom}, R_{Skmed}, $R_{Sqarith}$, R_{Sqmax}. and R_{Smax}.Then compute the corresponding total mean squared error for the above case respectively.
e. Repeat this process 1,000 times and then calculate the average of the mean squared error and the (PR) for each ridge parameter R, under consideration. Values of total mean square errors and PR are given in Tables 1-10.

3.2 Factors

In designing the Monte Carlo experiment we relied on a number of factors to evaluate the performance of the proposed SUR ridge regression estimators $\hat{\alpha}_{\mathrm{SUR}}(\mathrm{R})$ and to compare them with the SGLS estimator $\hat{\alpha}$. In this section we present these factors and values over which these factors were allowed to vary.

3.2.1. Number of equations. Our primary interest lies in the analysis of system-wise estimation, and thus the number of equations, M, to be estimated is of central importance. For example, as M increases the computation time becomes larger and lager. For this we considered M = 10 and 3 to designate large and small models respectively.

3.2.2. Number of observations per equation. To investigate the effect of sample size, T, on the properties of the suggested SUR-ridge parameters, we set T to 30 and 100 observations.

3.2.3. True value of the regression coefficient B. For B = (1,1,1,1,1)' and B = (1,2,3,4,5)' the TMSE values are listed in Tables 1-6 and 7-10, respectively. Clearly the TMSE values listed in Tables 7-10 are significantly larger than the corresponding TMSE values presented in Tables 1-4.

3.2.4. Distribution of X and collinearity among columns of X. Another factor that may effect the performance of the suggested SUR-ridge parameters is the strength and type of dependency among the explanatory variables. The explanatory variables were generated from a multivariate normal distribution, $MVN_4(0, \Sigma_x)$, a multivariate T-distribution, T(2), or T(6) respectively. The variance-covariance matrix Σ_x is defined as $\mathrm{diag}(\Sigma_x) = 1$ and off-$\mathrm{diag}(\Sigma_x)= \rho_x$. The strength of collinearity among these variables took on these values $\rho_x = 0.75, 0.90,$ 0.97 and 0.99, (for medium, high and very high).

3.2.5. Distribution of random errors and correlation among equations. The random errors were generate from a multivariate normal distribution $MVN_m(0, \Sigma_e)$, where m= 3 or 10 equations. The variance covariance matrix Σ_x is defined as $\mathrm{diag}(\Sigma_e) = 1$ and off-$\mathrm{diag}(\Sigma_e) = \rho_\Sigma$. Two different degrees of interdependency among these equations were considered. These values are $\rho_{\Sigma} = 0.35$ and 0.75, for low and high interdependency respectively.

Results with other values and combinations of these factors are available from the authors on request.

4. Results

In this section we analyze the output from the Monte Carlo experiment along with the main dominating factors effecting the properties of the different multivariate ridge parameters. In Tables 1-6 and 7-10 we display the TMSE and PR values for the proposed SUR ridge parameters R when the true values of B took on these values (1, 1, 1, 1, 1)' and (1,2,3,4,5)'. It is noticed that the TMSE values displayed in Tables 7-10 are significantly larger than the corresponding TMSE values in Tables 1-4 with some difference with regard to other properties. For this, the number of values of ρ_{X} considered in Tables 7-10 to the required number for this investigation, namely 0.75 (medium) and 0.97 (high).

It is noticed that the properties of SUR ridge estimators $\hat{\alpha}_{SUR}$ based on generating the matrix X from T(df), with df $\geq$ 6 are close to those of SUR ridge estimators based on generating the X matrix from a multivariate normal distributions. For this, values of the factor ρ_{X}, in Tables 5-6, are reduced to needed values for this study. These values are $\rho_{X} = 0.75$ and 0.97.

Moreover, our analysis has revealed four factors that have a bearing on the performance of the multivariate ridge parameters in terms of TMSE and PR criterion. These factors are the number of equations (M), number of observations (T), correlation among the explanatory variables (ρ_{X}) and correlation among equations (ρ_{Σ}). In all cases the performance of the multivariate ridge parameters, in terms of the above factors, can be summarized as

a. As M increases the TMSE increases and PR increases.

b. As T increases the TMSE decreases and PR increases.

c. As ρ_X increases the TMSE increases and PR decreases.

d. As ρ_Σ increases the TMSE increases and PR increases.

In addition, the output presented in Tables 1-10 shows a slight increase in the TMSE values For $\hat{\alpha}(R_{Sarith})$, $\hat{\alpha}(R_{Sqarith})$ and $\hat{\alpha}(R_{Smax})$ as the sample size increases. Theses multivariate ridge regression estimators have shown to have the best performance in terms of TMSE and PR criterion when compared with the remaining proposed multivariate ridge regression estimators. Moreover, multivariate ridge regression estimators based on R_{SK}, R_{SHK} and R_{Sharm} have produced the highest TMSE and the worst PR values among other estimators. In large samples when the correlation among the explanatory variables is low the unbiased estimator $\hat{\alpha}$, SGLS of α, has occasionally shown to have the smallest TMSE among the remaining estimators.

It is noticed that the TMSEs of almost all of the different parameters have enormously increased when the dimension of the system of equations is raised to 10 (the TMSEs are between 60 to 50,000), except for the SUR regression estimators based on R_{SK}, R_{SHK} and R_{Sharm} for which the increase in TMSE is not that dramatic (the TMSEs are between 60 to 22350). The SK, SHK and Sharm have produced the highest TMSE and the worst proportion of replications,(PR), among the others. In this case we suggest that there is no gain in efficiency for these two parameters when performing system-wise estimation.

5. Summary and conclusions

This paper has suggested a number of procedures to develop some new multivariate biased estimators applicable to systems of regression equations. All in all, 10 multivariate parameters are studied and compared. This investigation used the TMSE and the PR criterion to measure the goodness of SUR ridge-type estimators. The simulation results support the hypothesis that the number of equations, the number of observations per equation, the correlation among explanatory variables and equations are the main factors that affect the properties of SUR ridge estimators. It is noticed that the unbiased estimator, SGLS, has occasionally (in large sample and low correlation among explanatory variables) shown to have the smallest TMSE when compared with the others. However for high correlation, ρ_x, SUR ridge estimators based on R_{Sarith}, $R_{Sqarith}$ and R_{Smax} perform better than the remaining estimators, in particular $\hat{\alpha}(R_{Smax})$. It is evident from Tables 3-4, that the estimators $\hat{\alpha}(R_{Smax})$ performs quite well under all conditions or combination of factors discussed earlier. Clearly, SUR ridge estimators based on R_{SK}, R_{SHK} and R_{Sharm} perform very poorly when compared to the other estimators.

In conclusion, under certain conditions we may suggest the S_{max} as one of the good estimators to estimate the multivariate ridge parameter R. However, this requires further considerations such as generating random errors from some non-normal distribution.

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Appendices. Proofs of previous Results and Statements.

The following statements and results will prove useful.

Appendix A. Multivariate T distribution.

Given

$$f(x \mid n, \mu, \Sigma) = c[1 + \frac{1}{n}(x - \mu)'\Sigma^{-1}(x - \mu)], \text{ where } c = \frac{\Gamma(\frac{n+k}{2}) \mid \Sigma \mid^{-1/2}}{(n\pi)^{k/2}\Gamma(\frac{n}{2})}.$$

Then f is said to be the pdf for a k-dimensional multivariate T with n degrees of freedom, unknown location vector parameter $\mu \in R^k$ and covariance matrix Σ. To draw $\mathbf{x} \in \mathbf{R}^k$ from the above pdf,

1. Generate kx1 vector $Y \sim N_k(0, \Sigma)$.
2. Generate $Z \sim \chi^2(n)$ independent of Y.
3. Let the random independent vector $X' = (X_1, X_2, \cdots, X_k)$ be defined as

$$X_i = Y_i(\frac{Z}{n})^{-1/2} + \mu_i, \quad i = 1, 2, \cdots, k,$$

where $\mu' = (\mu_1, \mu_2, \cdots, \mu_k) \in R^k$ and $Y' = (Y_1, Y_2, \cdots, Y_k)$.
(See Koop, 2003).

1. Show that

a. $(\Lambda + R)^{-1} R\alpha\alpha' R(\Lambda + R)^{-1} = [(\Lambda + R)^{-1}\Lambda - I]\alpha\alpha'[(\Lambda + R)^{-1}\Lambda - I]$

b. $MSE(\hat{\alpha}_{SUR}) = (\Lambda + R)^{-1}(\Lambda + R\alpha\alpha' R')(\Lambda + R)^{-1}$

proof.

a. $-R = \Lambda - (\Lambda + R) \Rightarrow -(\Lambda + R)^{-1} R\alpha = [(\Lambda + R)^{-1}\Lambda - I]\alpha \Rightarrow$

$$(\Lambda + R)^{-1} R\alpha\alpha' R(\Lambda + R)^{-1} = [(\Lambda + R)^{-1}\Lambda - I]\alpha\alpha'[(\Lambda + R)^{-1}\Lambda - I]$$

b. $\hat{\alpha}_{SUR} = (Z'Z + R)^{-1} Z'Y^* = (Z'Z + R)^{-1} Z'Z\alpha + (Z'Z + R)^{-1} Z'e^* \Rightarrow$

$\hat{\alpha}_{SUR} - \alpha = -(Z'Z + R)^{-1} R\alpha + (Z'Z + R)^{-1} Z'e^*$. This leads to

$$E(\hat{\alpha}_{SUR} - \alpha)(\hat{\alpha}_{SUR} - \alpha)' = (Z'Z + R)^{-1}(Z'Z + R\alpha\alpha' R')(Z'Z + R)^{-1}$$
$$= (\Lambda + R)^{-1}(\Lambda + R\alpha\alpha' R')(\Lambda + R)^{-1}$$

2. Proof of Result 1.

a. $\text{MSE}(\hat{\alpha}) - \text{MSE}(\hat{\alpha}_{\text{SUR}}) = \Lambda^{-1} - (\Lambda + R)^{-1}(\Lambda + R\alpha\alpha' R')(\Lambda + R)^{-1}$

$= (\Lambda + R)^{-1}((\Lambda + R)^{-1}\Lambda^{-1}(\Lambda + R)^{-1} - (\Lambda + R\alpha\alpha' R'))(\Lambda + R)^{-1}$

$= (\Lambda + R)^{-1}(2R + R\Lambda^{-1}R - R\alpha\alpha' R')(\Lambda + R)^{-1}$

$= (\Lambda + R)^{-1}R(2R^{-1} + \Lambda^{-1} - \alpha\alpha')R(\Lambda + R)^{-1}$ is positive semidefinite matrix iff

$2R^{-1} + \Lambda^{-1} - \alpha\alpha'$ is positive semidefinite matrix iff for all $\delta \neq 0$,

$\delta'(2R^{-1} + \Lambda^{-1} - \alpha\alpha')\delta \geq 0$ or $\dfrac{(\delta'\alpha)^2}{\delta'(2R^{-1} + \Lambda^{-1})\delta} \leq 1$ which is true for all $\delta \neq 0$

if the maximum value of LHS of the above expression is less or equal to unity for all δ, $\delta \neq 0$. This leads to the condition $\alpha'(\Lambda^{-1} + 2R^{-1})\alpha \leq 1$, (see Rao, 1973, p 60). The interested reader may use the expression in (2.8), (Rao, 1973, p. 33) to confirm part b. And Result 1(c) follows Result 1b(ii).

3. Show that

$\text{var}(\hat{\alpha})$ exceeds $\text{var}(\hat{\alpha}_{\text{SUR}})$ by a positive semi-definite matrix.

proof. $\text{var}(\hat{\alpha}) - \text{var}(\hat{\alpha}_{\text{SUR}}) = \Lambda^{-1} - (\Lambda + R)^{-1}\Lambda(\Lambda + R)^{-1}$

$= (\Lambda + R)^{-1}[(\Lambda + R)\Lambda^{-1}(\Lambda + R) - \Lambda](\Lambda + R)^{-1}$

$= (\Lambda + R)^{-1}R[2R^{-1} + \Lambda^{-1}]R(\Lambda + R)^{-1} \geq 0$ is a positive semi-definite matrix.

TABLE 1. System-wise estimated TMSEs for the different methods, M =3 equations (Normal).

T = 30, ρ_Σ = 0.35										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	25.31	53.15 (99.8)	54.22 (99.8)	50.18 (99.7)	33.29 (79.5)	39.82 (96.2)	41.73 (97.1)	33.09 (79.2)	37.87 (91.6)	33.49 (79.5)
0.90	49.89	95.54 (100)	98.32 (100)	86.95 (99.9)	37.81 (33.0)	55.13 (70.3)	60.05 (77.6)	39.29 (31.4)	45.88 (49.2)	37.39 (33.6)
0.97	145.00	261.42 (98.9)	271.18 (99.2)	228.71 (96.9)	40.63 (2.6)	88.76 (6.2)	100.55 (17.5)	39.47 (2.5)	50.79 (3.1)	40.07 (2.6)
0.99	416.54	737.08 (96.7)	766.95 (98.4)	634.85 (88.6)	41.44 (0.0)	143.92 (0.1)	182.27 (3.3)	40.30 (0.0)	49.45 (0.0)	41.53 (0.0)
T = 100, ρ_Σ = 0.35										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	14.49	40.93 (99.9)	41.32 (99.9)	40.45 (99.9)	37.84 (99.9)	38.34 (99.9)	38.83 (99.9)	37.19 (99.9)	38.47 (99.9)	39.18 (99.9)
0.90	20.89	54.20 (100.0)	55.10 (100.0)	52.18 (100.0)	40.68 (97.9)	44.82 (99.9)	46.47 (99.9)	40.21 (98.0)	43.83 (99.8)	41.66 (98.0)
0.97	45.67	96.66 (100.0)	99.61 (100.0)	88.65 (100.0)	42.84 (49.7)	57.39 (83.6)	63.25 (90.2)	42.20 (48.6)	49.23 (64.1)	43.43 (50.9)
0.99	116.42	216.51 (99.8)	225.48 (99.9)	190.01 (99.3)	43.49 (5.2)	77.67 (10.2)	93.15 (27.6)	42.62 (4.6)	50.42 (5.3)	44.25 (5.9)

T = 30, ρ_Σ = 0.75										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	26.39	55.05 (99.8)	55.73 (99.8)	52.89 (99.8)	36.33 (84.6)	43.49 (97.9)	44.44 (98.2)	36.69 (85.9)	42.09 (95.3)	35.43 (82.4)
0.90	50.19	99.16 (99.8)	100.94 (99.8)	93.09 (99.6)	41.13 (37.7)	62.23 (81.6)	65.12 (83.2)	41.30 (38.0)	52.77 (62.2)	39.34 (34.8)
0.97	142.20	262.65 (99.0)	268.87 (99.3)	240.51 (98.1)	44.13 (2.5)	107.26 (21.8)	121.26 (36.2)	43.36 (2.4)	61.40 (4.6)	42.07 (2.8)
0.99	404.89	720.38 (96.7)	739.31 (97.4)	654.43 (93.2)	44.55 (0.2)	188.39 (0.8)	244.52 (13.0)	43.35 (0.2)	59.96 (0.2)	43.23 (0.2)
T = 100, ρ_Σ = 0.75										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	16.45	43.62 (100.0)	43.82 (100.0)	43.31 (100.0)	40.57 (99.8)	41.75 (99.9)	42.17 (99.9)	40.46 (99.8)	41.80 (99.9)	40.94 (99.9)
0.90	22.93	58.90 (99.8)	59.41 (99.8)	57.68 (99.8)	45.00 (99.7)	51.46 (99.8)	52.66 (99.7)	45.34 (99.1)	50.31 (99.7)	44.73 (97.7)
0.97	47.98	107.07 (100.0)	108.81 (100.0)	101.93 (100.0)	47.60 (54.3)	71.01 (95.5)	76.54 (96.2)	47.70 (54.6)	59.64 (78.1)	46.75 (52.1)
0.99	119.50	237.40 (99.7)	242.70 (99.7)	221.25 (99.5)	48.09 ()5.9	107.56 (37.7)	127.15 (60.9)	47.62 (5.7)	64.29 (8.8)	47.34 (6.1)

The PR values (in percentage) are placed in parenthesis below the values of the corresponding estimators.

TABLE 2. Estimated TMSEs and PRS for the different methods, M = 10 equations (Normal).

T = 30, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	132.56	290.16 (100.0)	295.09 (100.0)	271.28 (100.0)	133.44 (52.6)	198.44 (99.5)	209.72 (99.8)	133.91 (53.2)	171.70 (89.1)	131.48 (49.5)
0.90	283.01	599.89 (100.0)	612.44 (100.0)	551.99 (100.0)	142.29 (0.8)	311.63 (72.9)	341.25 (85.3)	141.99 (0.9)	206.04 (11.9)	139.20 (0.7)
0.97	865.72	185.65 (100.0)	1894.85 (100.0)	1681.28 (99.8)	160.57 (0.0)	525.78 4.9()	745.39 (25.7)	158.69 (0.0)	238.05 (0.0)	158.96 (0.0)
0.99	2529.78	5494.19 (99.9)	5625.06 (99.9)	4954.89 (99.7)	165.89 (0.0)	1197.87 (0.0)	1629.72 (7.3)	164.15 (0.0)	228.02 (0.0)	166.69 (0.0)

T = 100, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	61.87	177.66 (100.0)	179.26 (100.0)	173.09 (100.0)	153.04 (100.0)	158.01 (100.0)	160.52 (100.0)	151.19 (100.0)	157.49 (100.0)	155.86 (100.0)
0.90	94.24	253.57 (100.0)	257.74 (100.0)	240.65 (100.0)	165.66 (99.4)	189.47 (100.0)	197.57 (100.0)	163.87 (99.2)	179.39 (99.8)	169.09 (99.4)
0.97	219.50	534.33 (100.0)	548.46 (100.0)	486.05 (100.0)	185.10 (28.7)	259.20 (85.6)	290.85 (95.6)	182.54 (26.9)	205.38 (41.0)	190.10 (31.8)
0.99	577.18	1272.14 (100.0)	1314.77 (100.0)	1125.88 (99.9)	190.09 (0.1)	353.45 (0.3)	452.80 (13.2)	187.28 (0.1)	205.65 (0.1)	195.66 (0.1)

T = 30, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	131.62	309.43 (100.0)	312.04 (100.0)	298.52 (100.0)	157.84 (78.2)	238.26 (100.0)	248.66 (100.0)	159.93 (80.3)	210.99 (98.6)	151.93 (71.1)
0.90	278.45	625.13 (99.9)	632.12 (99.0)	595.55 (99.9)	163.59 (3.2)	385.21 (96.9)	413.96 (98.3)	165.01 (3.1)	265.82 (46.5)	153.99 (2.2)
0.97	847.05	1833.91 (100.0)	1857.96 (100.0)	1735.92 (100.0)	166.92 (0.0)	822.39 (40.7)	947.47 (65.4)	166.11 (0.0)	318.05 (0.0)	158.05 (0.0)
0.99	2470.78	5448.49 (99.8)	5521.43 (99.9)	5152.42 (99.7)	181.31 (0.0)	1772.08 (7.2)	2286.40 (36.2)	179.50 (0.0)	328.34 (0.0)	176.84 (0.0)

T = 100, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	63.10	203.26 (100.0)	204.04 (100.0)	201.14 (100.0)	182.25 (100.0)	190.70 (100.0)	192.73 (100.0)	181.67 (100.0)	190.21 (100.0)	183.21 (100.0)
0.90	95.16	282.82 (100.0)	284.86 (100.0)	275.71 (100.0)	192.62 (100.0)	232.66 (100.0)	240.09 (100.0)	192.60 (100.0)	220.47 (100.0)	193.67 (100.0)
0.97	219.25	530.41 (100.0)	537.34 (100.0)	505.79 (100.0)	178.56 (24.6)	313.12 (99.2)	345.29 (99.7)	178.64 (23.9)	232.41 (62.6)	178.81 (24.5)
0.99	573.58	1296.63 (100.0)	1317.74 (100.0)	1223.63 (100.0)	202.61 (0.0)	532.80 (30.3)	652.17 (70.8)	201.15 (0.0)	261.26 (0.1)	205.13 (0.0)

The PR values (in percentage) are placed in parenthesis below the values of the corresponding estimators.

TABLE 3. System-wise estimated TMSEs and PR for the different methods, M =3 equations, T(2).

T = 30, ρ_Σ = 0.35										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	109.55	154.05 (97.0)	155.09 (97.3)	150.15 (96.2)	37.48 (50.2)	109.46 (74.4)	113.93 (78.2)	39.37 (51.2)	63.02 (64.9)	33.21 (48.5)
0.90	271.56	381.65 (98.0)	384.36 (98.4)	369.85 (95.5)	44.81 (25.0)	212.84 (52.4)	231.44 (57.4)	45.78 (24.4)	82.00 (35.9)	37.81 (23.5)
0.97	898.65	1414.34 (95.6)	1423.72 (97.5)	1370.78 (91.3)	59.61 (3.8)	540.78 (18.0)	493.29 (23.4)	54.50 (3.7)	117.68 (6.1)	42.49 (3.7)
0.99	2689.28	3764.09 (92.7)	3792.72 (95.0)	3629.57 (83.2)	59.95 (0.1)	1102.21 (4.9)	953.31 (9.3)	52.02 (0.1)	128.21 (0.3)	43.79 (o.2)
T = 100, ρ_Σ = 0.35										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	86.72	131.73 (99.1)	132.07 (99.1)	130.94 (99.1)	36.53 (86.9)	122.22 (96.0)	124.85 (96.8)	39.43 (87.3)	53.97 (92.6)	36.35 (86.1)
0.90	216.03	427.21 (99.4)	428.21 (99.5)	424.38 (99.4)	41.28 (73.2)	368.93 (89.7)	395.73 (91.9)	44.35 (74.3)	69.34 (83.3)	40.10 (72.6)
0.97	716.41	1707.49 (98.4)	1710.29 (98.4)	1696.38 (07.8)	45.88 (32.7)	1272.57 (57.4)	1546.07 (61.4)	47.92 (32.6)	92.97 (42.5)	42.30 (32.9)
0.99	2145.16	5132.32 (97.5)	5140.85 (97.7)	5096.04 (93.0)	67.86 (6.4)	3655.04 (17.9)	4490.23 (25.5)	57.46 (6.1)	142.68 (7.8)	46.25 (7.0)

T = 30, ρ_Σ = 0.75										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	113.75	190.57 (98)	191.20 (98)	188.09 (97.8)	41.65 (54.5)	151.10 (83.2)	159.75 (83.9)	44.72 (56.0)	70.15 (72.3)	36.09 (50.5)
0.90	282.16	463.45 (97.9)	465.09 (98.0)	456.15 (97.5)	54.75 (28.4)	339.34 (66.2)	352.55 (66.3)	56.42 (27.5)	105.45 (45.7)	41.42 (25.1)
0.97	933.75	1310.35 (96.3)	1316.00 (97.0)	1282.48 94.1()	72.29 (5.7)	820.48 (30.7)	818.11 (37.3)	65.88 (5.4)	167.84 (10.8)	45.93 (4.8)
0.99	2794.25	4213.37 (94.8)	4239.65 (95.3)	4130.65 (90.0)	79.20 (0.4)	2322.33 (11.6)	2013.20 (15.8)	63.18 (0.3)	180.30 (0.6)	47.53 (0.4)
T = 100, ρ_Σ = 0.75										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	83.71	184.50 (99.6)	184.68 (99.6)	184.03 (99.6)	42.51 (87.6)	176.62 (97.8)	179.49 (98.3)	49.25 (89.7)	71.18 (94.8)	39.41 (86.8)
0.90	201.78	310.27 (99.4)	310.74 (99.4)	308.61 (99.4)	50.00 (75.3)	279.15 (95.2)	284.76 (95.4)	57.98 (77.7)	99.81 (89.0)	44.00 (72.2)
0.97	658.05	1103.05 (98.2)	1104.67 (98.3)	1096.71 (97.9)	59.91 (35.8)	958.27 (74.3)	983.56 (73.7)	67.97 (35.6)	170.21 (53.0)	46.92 (32.2)
0.99	1960.66	2998.51 (96.8)	3003.42 (97.3)	2978.52 (95.3)	119.26 (5.6)	2485.17 (37.5)	2500.74 (41.6)	93.64 (5.1)	269.72 (12.8)	57.14 (5.4)

The PR values (in percentage) are placed in parenthesis below the values of the corresponding estimators.

TABLE 4. Estimated TMSEs and PRs for the different methods, M = 10 equations, T(2)

T = 30, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	745.87	1386.51 (96.9)	1391.17 (97.1)	1367.70 (96.5)	172.41 (5.9)	1105.46 (75.7)	1130.30 (81.7)	174.94 (5.7)	509.93 (38.2)	129.15 (4.0)
0.90	1937.55	3178.82 (97.8)	3190.73 (97.9)	3123.10 (07.1)	216.07 (0.0)	2346.60 (49.3)	2420.95 (59.0)	202.28 (0.0)	745.35 (5.0)	140.54 (0.0)
0.97	6552.52	12730.1 (99.0)	12771.2 (99.0)	12519.8 (98.6)	263.49 (0.0)	8311.10 (20.9)	8540.64 (31.1)	211.89 (0.0)	1172.73 (0.0)	153.53 (0.0)
0.99	19731.4	36585.7 (99.0)	36710.3 (99.2)	35920.1 (98.8)	283.16 (0.0)	20760.5 (10.9)	21546.8 (17.2)	209.22 (0.0)	1305.83 (0.0)	163.74 (0.0)

T = 100, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	208.25	402.09 (99.5)	403.57 (99.5)	397.27 (99.5)	156.42 (76.2)	335.52 (96.1)	348.73 (97.4)	162.67 (77.1)	257.03 (90.2)	154.64 (75.5)
0.90	485.51	839.78 (98.7)	843.68 (98.8)	824.91 (98.3)	171.81 (38.7)	627.59 (84.7)	661.03 (87.8)	180.98 (39.2)	363.60 (64.9)	163.95 (39.4)
0.97	1557.64	2795.00 (99.2)	2808.74 (99.2)	2737.93 (99.1)	197.32 (1.4)	1680.67 (33.6)	1805.30 (44.6)	202.00 (1.4)	539.55 (5.5)	117.34 (1.7)
0.99	4618.70	8077.39 (99.4)	8119.60 (99.4)	7892.40 (99.2)	212.02 (0.0)	4161.00 (6.2)	4441.86 (13.2)	210.02 (0.0)	656.81 (0.0)	186.25 (0.0)

T = 30, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	1096.77	1743.12 (97.7)	1745.51 (97.7)	1732.38 (97.3)	263.56 (14.2)	1588.90 (91.6)	1616.70 (93.3)	281.09 (14.2)	946.39 (64.8)	153.90 (7.8)
0.90	2929.46	4459.10 (98.3)	4465.57 (98.3)	4428.75 (98.1)	373.62 (0.9)	3859.91 (76.3)	3941.18 (82.0)	341.73 (0.6)	1627.85 (18.5)	167.97 (0.5)
0.97	10036.8	17990.4 (98.8)	18012.8 (98.8)	17873.4 (98.6)	1039.92 (0.0)	14615.0 (48.1)	15041.0 (58.1)	530.00 (0.0)	3424.89 (0.2)	252.64 (0.0)
0.99	30336.5	48478.5 (98.9)	48546.7 (98.9)	48128.8 (98.2)	1635.43 (0.0)	36591.7 (27.9)	37358.4 (41.7)	782.96 (0.0)	5506.74 (0.0)	284.75 (0.0)

T = 100, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	215.34	415.65 (99.4)	416.28 (99.4)	413.39 (99.4)	189.18 (83.1)	380.11 (98.8)	387.47 (98.9)	207.24 (85.0)	322.81 (96.5)	176.27 (80.5)
0.90	502.92	962.91 (99.5)	964.69 (99.5)	955.97 (99.4)	213.38 (50.3)	823.44 (96.1)	844.39 (96.5)	237.28 (51.9)	549.28 (82.8)	186.42 (47.5)
0.97	1615.10	2994.24 (99.1)	3000.65 (99.1)	2966.35 (99.1)	235.89 (1.8)	2243.27 (72.1)	2328.69 (76.1)	251.75 (1.8)	913.76 (18.3)	186.74 (1.7)
0.99	4790.60	8416.67 (99.5)	8436.51 (99.5)	8330.57 (99.5)	275.45 (0.0)	5580.19 (29.9)	5713.38 (41.7)	258.04 (0.0)	1304.36 (0.1)	194.48 (0.0)

The PR values (in percentage) are placed in parenthesis below the values of the corresponding estimators.

TABLE 5. Estimated TMSEs and PRs for the different methods, M =3 equations, T(6).

T = 30, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	32.12	61.49 (99.6)	62.53 (99.7)	58.19 (99.3)	33.72 (67.3)	44.42 (88.7)	46.34 (91.1)	33.64 (67.3)	40.82 (81.5)	33.17 (64.1)
0.97	205.93	359.06 (98.1)	368.57 (98.7)	321.52 (94.4)	42.29 (2.5)	116.67 (11.9)	125.99 (21.5)	40.47 (2.5)	58.19 (3.8)	39.81 (2.5)

T = 100, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	15.97	41.82 (100.0)	42.17 (100.0)	41.21 (100.0)	36.71 (99.0)	38.20 (99.8)	38.89 (99.9)	36.25 (99.9)	38.17 (99.6)	37.90 (98.9)
0.97	59.99	119.54 (99.7)	122.39 (99.7)	109.91 (99.4)	42.15 (41.1)	61.96 (68.2)	66.52 (73.5)	41.61 (39.6)	50.36 (52.1)	42.65 (41.8)

T = 30, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	32.83	61.97 (98.9)	62.62 (99.2)	59.83 (98.9)	37.19 (72.8)	48.27 (92.7)	49.45 (93.4)	37.80 (74.4)	45.27 (87.5)	35.58 (69.2)
0.97	198.31	344.83 ()98.6	350.71 (99.1)	320.65 (96.5)	47.85 (4.5)	139.29 (24.0)	150.63 (39.4)	45.73 (3.9)	71.86 (6.5)	42.20 (3.9)

T = 100, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	18.16	44.95 (100.0)	45.14 (100.0)	44.61 (100.0)	40.11 (98.8)	42.56 (99.8)	42.99 (99.8)	40.33 (99.0)	42.50 (99.8)	40.10 (98.6)
0.97	63.90	127.48 (99.6)	129.12 (99.7)	122.13 (99.6)	47.87 (44.1)	83.05 (86.3)	86.43 (87.3)	48.51 (43.9)	66.26 (66.9)	45.54 (41.9)

TABLE 6. Estimated TMSEs and PRs for the different methods, M = 10 equations, T(6).

T = 30, ρ_Σ = 0.35										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	174.97	344.22 (100.0)	348.95 (100.0)	326.11 (99.9)	135.21 (26.1)	234.11 (93.7)	247.35 (96.7)	135.84 (25.9)	191.55 (69.3)	130.48 (21.6)
0.97	1261.04	2453.99 (99.8)	2495.48 (99.9)	2274.60 (99.6)	159.53 (0.0)	878.79 (6.4)	1013.17 (21.1)	157.23 (0.0)	285.01 (0.0)	154.07 (0.0)
T = 100, ρ_Σ = 0.35										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	70.91	185.25 (100.0)	186.74 (100.0)	181.33 (100.0)	151.49 (99.7)	162.70 (100.0)	165.76 (100.0)	150.16 (99.7)	160.86 (100.0)	153.97 (99.7)
0.97	304.20	635.64 (100.0)	648.90 (100.0)	591.65 (100.0)	178.25 (9.8)	300.13 (57.2)	339.36 (74.1)	176.41 (9.1)	214.79 (19.0)	182.38 (11.5)

T = 30, ρ_Σ = 0.75										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	174.87	362.36 (100.0)	364.79 (100.0)	352.11 (100.0)	161.64 (46.5)	282.10 (98.8)	293.70 (99.2)	164.50 (48.9)	238.57 (89.9)	150.26 (34.4)
0.97	1250.63	2490.84 (99.7)	2513.52 (99.7)	2389.66 (99.7)	173.75 (0.0)	1184.04 (39.6)	1184.04 (54.9)	1316.96 (0.0)	171.62 (0.0)	157.56 (0.0)
T = 100, ρ_Σ = 0.75										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	71.68	209.02 (100.0)	209.72 (100.0)	207.14 (100.0)	178.46 (99.9)	194.72 (100.0)	197.16 (100.0)	179.02 (99.9)	193.07 (100.0)	178.80 (99.9)
0.97	302.78	643.73 (99.9)	650.19 100.0()	620.74 (99.9)	180.76 (9.7)	380.09 (88.1)	418.22 (91.4)	181.89 (9.6)	263.46 (39.4)	180.27 (9.8)

The PR values (in percentage) are placed in parenthesis below the values of the corresponding estimators.

TABLE 7. Estimated TMSEs and PRs for the different methods, M =3 equations, Normal.

T = 30, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	309.66	656.84 (99.9)	658.60 (99.9)	652.61 (99.9)	469.63 (90.5)	610.60 (99.8)	618.20 (99.9)	439.89 (87.3)	530.42 (98.0)	406.82 (79.7)
0.97	1676.81	3225.27 (99.6)	3240.86 (99.6)	3181.37 (99.5)	767.85 (10.2)	2386.40 (93.0)	2386.14 (90.7)	547.82 (3.6)	954.54 (11.8)	501.80 (3.4)

T = 100, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	186.60	509.89 (99.9)	509.89 (99.9)	508.47 (99.9)	468.26 (99.8)	501.10 (99.9)	503.28 (99.9)	464.94 (99.7)	489.26 (99.9)	454.35 (99.8)
0.97	541.76	1198.21 (100.0)	1202.86 (100.0)	1187.14 (100.0)	614.86 (66.8)	1035.68 (100.0)	1060.28 (100.0)	554.61 (56.8)	736.40 (87.6)	518.97 (50.7)

T = 30, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	333.72	687.35 (99.9)	688.44 (99.9)	684.52 (99.9)	515.06 (92.4)	651.90 (99.8)	654.91 (99.8)	490.53 (91.1)	584.63 (98.5)	443.62 (82.5)
0.97	1713.01.	3284.36 (99.5)	3293.26 (99.5)	3256.84 (99.4)	934.79 (15.0)	2663.14 (97)	2647.46 (94.2)	653.44 (3.8)	1226.72 (24.3)	559.63 (3.0)

T = 100, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	215.65	541.50 (99.9)	541.79 (99.9)	540.99 (99.9)	503.00 (99.4)	215.65 (99.9)	538.09 (99.9)	503.79 (99.5)	528.02 (99.9)	480.19 (99.2)
0.97	590.79	1324.97 (100.0)	1327.41 (100.0)	1318.99 (100.0)	729.41 (72.0)	1213.70 (100.0)	1224.37 (100.0)	658.00 (66.6)	915.11 (92.6)	557.42 (52.4)

TABLE 8. Estimated TMSEs and PRs for the different methods, M = 10 equations, Normal.

T = 30, ρ_Σ = 0.35										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	1641.59	3489.53 (100.0)	3496.24 (100.0)	3465.17 (100.0)	1856.55 (65.0)	3218.68 (100.0)	3268.58 (100.0)	1665.27 (53.1)	2523.40 (96.5)	1452.66 (30.6)
0.97	10337.8	22484.4 (100.0)	22538.8 (100.0)	22278.0 (100.0)	2654.00 (0.2)	17692.9 (99.4)	18296.4 (99.6)	1908.79 (0.0)	5145.86 (0.6)	1680.03 (0.0)
T = 100, ρ_Σ = 0.35										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	801.53	2011.53 (100.0)	2013.80 (100.0)	2004.81 (100.0)	1647.02 (99.9)	1947.45 (100.0)	1961.15 (100.0)	1636.95 (99.9)	1847.04 (100.0)	1573.59 (99.8)
0.97	2661.40	6277.35 (100.0)	6296.10 (100.0)	6214.17 (100.0)	1963.91 (13.9)	5106.11 (100.0)	5305.90 (100.0)	1867.16 (9.6)	2845.75 (64.5)	1795.41 (7.6)

T = 30, ρ_Σ = 0.75										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	1656.12	3576.98 (100.0)	3580.32 (100.0)	3564.40 (100.0)	2056.18 (73)	3407.07 (100.0)	3439.49 (100.0)	1853.10 (68.3)	2834.49 (98.7)	1516.11 (35.0)
0.97	10296.2	22440.4 (100.0)	22469.0 (100.0)	22331.7 (100.0)	3524.98 (2.0)	19473.5 (99.9)	19973.8 (99.9)	2273.21 (0.0)	7323.74 (13.5)	1761.73 (0.0)
T = 100, ρ_Σ = 0.75										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	824.12	2083.11 (100.0)	2084.04 (100.0)	2080.26 (100.0)	1766.88 (100.0)	2053.30 (100.0)	2060.29 (100.0)	1762.77 (100.0)	1986.55 (100.0)	1639.85 (100.0)
0.97	2708.47	6340.80 (100.0)	6349.35 (100.0)	6312.95 (100.0)	2261.06 (26.6)	5730.07 (100.0)	5843.67 (100.0)	2070.49 (16.0)	3714.56 (89.6)	1835.55 (7.9)

The PR values (in percentage) are placed in parenthesis below the values of the corresponding estimators.

TABLE 9. Estimated TMSEs and PRs for the different methods, M = 3 equations, T(2).

T = 30, ρ_Σ = 0.35										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	1348.26	1895.70 (98.0)	1897.29 (98.0)	1890.74 (97.9)	933.48 (66.6)	1787.44 (95.8)	1793.48 (95.7)	721.81 (60.0)	1276.23 (80.5)	530.46 (52.1)
0.97	10908.2	17283.9 (97.4)	17297.6 (97.4)	17228.3 (97.4)	2812.69 (13.6)	14636.3 (79.8)	13730.2 (78.0)	1219.82 (5.9)	3483.41 (18.3)	987.60 (4.8)
T = 100, ρ_Σ = 0.35										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	1122.5	1653.30 (99.0)	1653.76 (99.0)	1652.11 (99.0)	689.17 (93.3)	1635.44 (98.9)	1637.67 (98.9)	715.671 (92.5)	1297.22 (97.2)	529.92 (87.0)
0.97	8876.55	21944.1 (98.5)	21948.2 (98.5)	21929.6 (98.5)	1872.89 (46.9)	21390.2 (94.6)	21374.9 (93.1)	1294.98 (40.0)	3817.82 (62.3)	773.58 (35.4)

T = 30, ρ_Σ = 0.75										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	1408.98	2349.93 (97.9)	2350.89 (97.9)	2346.99 (97.9)	1098.06 (70.6)	2276.57 (97.1)	2286.18 (97.2)	871.78 (65.1)	1663.44 (87.7)	565.72 (53.7)
0.97	11382.1	16059.1 (97.3)	16066.9 (97.4)	16026.2 (97.2)	3813.56 (19.0)	14568.7 (87.6)	14291.0 (84.7)	1649.29 (8.3)	5523.76 (27.9)	1082.37 (6.3)
T = 100, ρ_Σ = 0.75										
ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	1081.03	2566.45 (99.5)	2566.70 (99.5)	2565.81 (99.5)	926.60 (94.6)	2555.52 (99.5)	2557.00 (99.5)	935.08 (94.8)	1861.04 (98.5)	681.23 (88.9)
0.97	8155.42	13470.4 (98.4)	13472.6 (98.5)	13463.0 (98.4)	3551.8 (56.1)	13225.2 (97.5)	13152.5 (96.8)	2610.3 (46.6)	6948.65 (76.5)	1645.78 (36.2)

TABLE 10. Estimated TMSEs and PRs for the different methods, M = 10 equations, T(2).

T = 30, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	9282.79	16894.3 (96.6)	16900.0 (96.6)	16873.9 (96.6)	4908.77 (25.6)	16392.2 (95.8)	16485.4 (96.1)	3197.49 (8.2)	9483.96 (67.9)	1723.19 (2.1)
0.97	80242.4	157573.2 (98.9)	157621.3 (98.9)	157342.2 (98.8)	25355.98 (1.4)	145433.3 (95.7)	145678.4 (95.3)	6382.18 (0.0)	39845.9 (3.3)	3453.25 (0.0)

T = 100, ρ_Σ = 0.35

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	2620.78	4718.44 (99.1)	4720.34 (99.1)	4712.87 (99.1)	2248.91 (73.6)	4589.32 (99.0)	4619.14 (98.9)	2099.98 (72.6)	3656.73 (94.6)	1510.61 (60.8)
0.97	19075.1	34002.0 (99.0)	34018.8 (99.1)	33938.3 (99.0)	4753.56 (2.0)	30939.4 (97.0)	31209.2 (97.4)	3182.11 (0.7)	12330.9 (21.8)	1894.96 (0.3)

T = 30, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	13906.0	21723.8 (96.7)	21726.8 (96.7)	21712.4 (96.7)	8968.13 (40.2)	21469.0 (96.4)	21511.3 (96.4)	5950.61 (19.4)	16451.0 (83.8)	2427.59 (3.6)
0.97	124027.6	223504.2 (98.5)	223529.2 (98.5)	223379.7 (98.5)	53795.0 (4.3)	216129.6 (97.6)	216897.4 (97.7)	13725.3 (0.0)	88067.7 (14.2)	6482.68 (0.0)

T = 100, ρ_Σ = 0.75

ρ_X	SGLS	SK	SHK	Sharm	Sarith	Sgeom	Skmed	Sqarith	Sqmax	Smax
0.75	2715.77	4632.02 (99.2)	4632.79 (99.2)	4629.57 (99.2)	2729.74 (80.7)	4576.82 (99.1)	4588.60 (99.1)	2647.89 (80.8)	4187.02 (97.2)	1697.15 (63.1)
0.97	19837.2	36379.2 (98.9)	36386.6 (98.9)	36350.1 (98.9)	8213.68 (7.4)	34833.1 (98.3)	34958.6 (98.5)	5149.24 (2.2)	18572.2 (55.7)	2280.74 (0.3)

The PR values (in percentage) are placed in parenthesis below the values of the corresponding estimators.