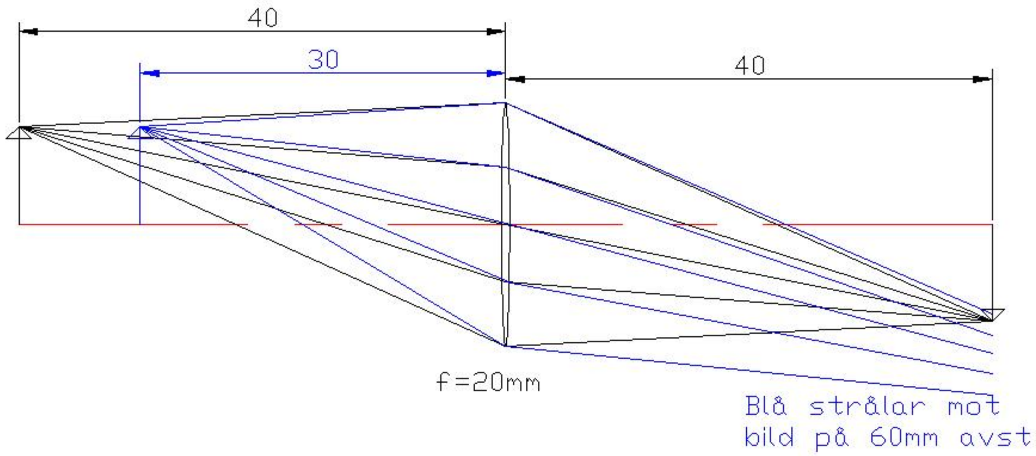
Telecentric imaging and perspective

In all kinds of imaging where measurements are to be performed on the image or corresponding, it is important to eliminate systematic sources of error. One such is that the magnification in an ordinary imaging depends non-linearly on the object distance

$$M = \frac{b}{a} = \frac{f}{a-f}$$

Another one arises if one wants to perform measurements on the background or the foreground, or on an object one does not know the exact distance to. The image becomes then diffuse, but (which is worse) the center of gravity of the diffuseness (centroid) will be located at another distance from the symmetry axis than the sharp image.
See the figure below, where the blue rays are propagating towards an object magnified by a factor of two, located at a distance of 60 mm behind the image recording surface, located in the image plane for a 1:1 imaging (object distance = image distance = 40 mm).



One can see clearly (?) that the blur generated by the blue rays is not centered at the true image.
Furthermore, longitudinal displacements on the object side give rise to a corresponding displacement on the image side, which is unfortunately not linear.
One can see above that a 10 mm displacement of the object will give a 20 mm displacement of the image and generally for small displacements we have that

$$(1) \qquad \Delta b = \frac{db}{da}\Delta a = \frac{d}{da}\left(\frac{af}{a-f}\right)\Delta a = \left[\frac{f}{a-f} - \frac{af}{(a-f)^2}\right]\Delta a = -\frac{f^2}{(a-f)^2}\Delta a = -M^2\Delta a$$

This is usually expressed so that the longitudinal magnification is the square of the transverse one (= the usual one), but does not have that much to do with magnification.

A solution to all these three problems is the so-called telecentric imaging, which contains two components:
An afocal system consisting of two lenses (lens system) with a common real focus in between
And
An aperture of such a size that it will become an aperture stop when placed in the common focus.
We start with the imaging in the afocal system and note that since it is afocal, one cannot use the principal plane in the calculation (why not?) but must calculate each lens separately.
The focal lengths of the lenses are f_1 and f_2 and the distance between them is thus $f_1 + f_2$, a_1 is the object distance to lens 1 etc.

$$(2) \quad b_1 = \frac{a_1 f_1}{a_1 - f_1} \quad \Rightarrow$$

$$(3) \quad a_2 = f_1 + f_2 - b_1 = \frac{a_1 f_2 - f_1^2 - f_1 f_2}{a_1 - f_1}$$

$$(4) \quad b_2 = \frac{a_2 f_2}{a_2 - f_2} = \frac{-f_2\left(a_1 f_2 - f_1 f_2 - f_1^2\right)}{f_1^2}$$

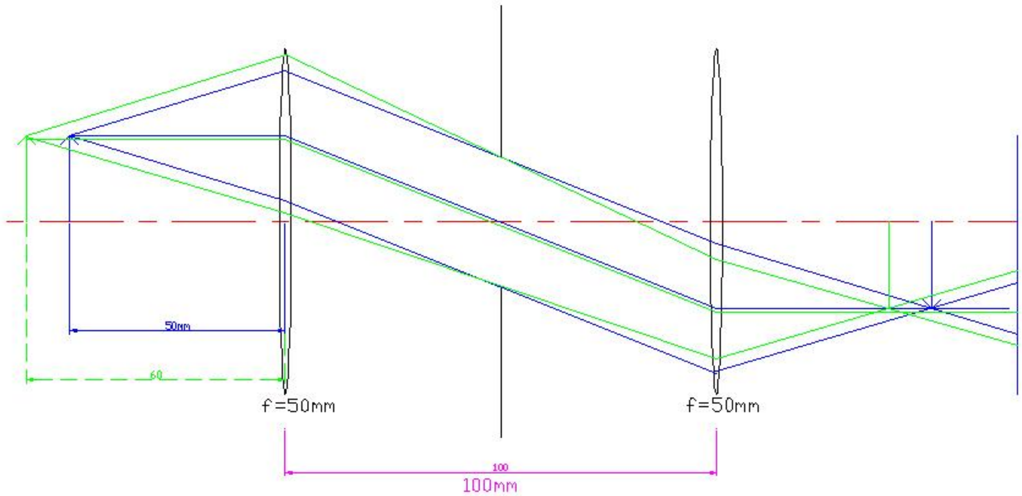
Now, it turns out to be practical to calculate at which distance behind the rear focus of the rear lens the image will appear, i.e.

$$(6) \quad f_2 - b_2 = \frac{f_2^2}{f_1^2}\left(a_1 - f_1\right)$$

Anybody who happens to know Newton’s variant of the lens formula gets his/her reward here.
This means that the distance between the image and the rear focus of the rear lens equals the distance between the object and the front focus of the front lens times the system’s telescopic magnification squared (!!)
Why is this so fun? Well, a given displacement of the object corresponds always to a certain displacement of the image, irrespective of the initial position of the object, i.e. the longitudinal magnification is always the same.
What happens with the ordinary (transverse) magnification?

$$(7) \quad M = \frac{b_1}{a_1}\frac{b_2}{b_1} = (a\,bit\,of\;work) = -\frac{f_2}{f_1}$$

i.e. the telescopic magnification does not depend on the position of the object.
And then we have this matter of the aperture:
If the aperture stop lies in the common focus, both the entrance and exit pupils will go to infinity, which means that the system will always accept rays within a given angular sector, irrespective of the object size.
A small figure, perhaps?

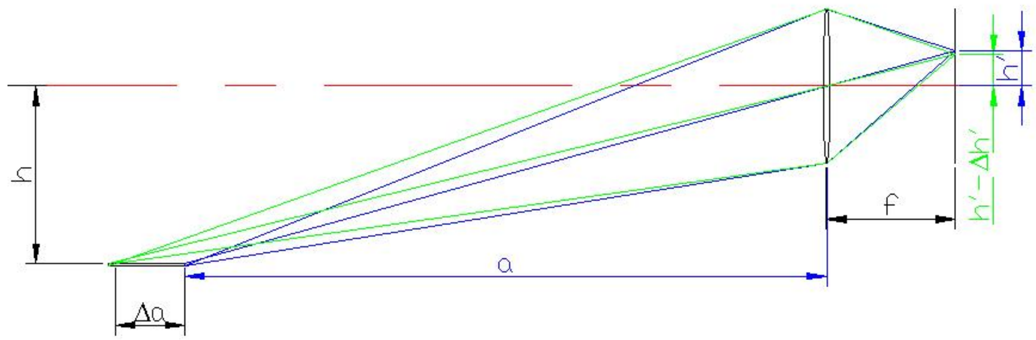


We have here two object positions, 50 mm and 60 mm in front of the first lens and we see that the displacements of the object and image are equal, since the lenses have equal focal length. The magnification is consequently equal to one.
The fact that the exit pupil is located at infinity means that the light cone going in towards or out from the image lies always symmetrically around the image, and the center of gravity ends up in the right place. Let for instance the image-recording surface be located in the blue image plane. The green light cone then has its axial center at the blue arrowhead.
This kind of imaging consequently solves all the problems lined up in the beginning and is mostly used in measurement on the image.
The drawbacks are that the system must be rather large. One can show that the lenses must be larger than the sum of object size, image size and the aperture size if one wants to be sure of avoiding vignetting at reasonable object distances. The vignetting destroys the entire reasoning above.
Try placing the object at 30 mm from the first lens in the above system… The next drawback is that it becomes rather expensive. Both lenses are in fact camera objectives, facing each other from the image side (why ?)

Perspective

When measuring in an image representing a three-dimensional reality, the longitudinal and transversal lengths will be magnified equally. The length of a stick lying on the symmetry axis can of course not be evaluated, which is possible for a

stick lying slightly sideways. Let us consider a stick with length Δa lying with its nearest point at a distance a from a normal (not telecentric) lens.
If we let the object distance be much larger than the focal length, the image distance will be = f and the transverse magnification consequently M = f/a.



We get

$$h' = \frac{fh}{a} \Rightarrow \Delta h' = -\frac{fh}{a^2}\Delta a = -\frac{f^2h}{a^2 f}\Delta a = \frac{-M^2h}{f}\Delta a = \frac{-Mh'}{f}\Delta a$$

The three last terms represent equivalent ways of writing the same thing.
Observe that this is consistent with what one was taught on perspective at drawing lessons in the primary school (?), namely on the construction lines converging towards a given point = extension of the optical axis to infinity.