Impedance: $j\omega$-method

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1 General properties

A complex (two-component) number has the form

$$c = a + jb$$

where $j = \sqrt{-1}$ is the *imaginary unit.* This form is equivalent to the polar form

$$Ae^{j\varphi} \quad \text{, with} \quad A = \sqrt{a^2 + b^2} \quad \text{and} \quad \varphi = \tan^{-1}\frac{b}{a}. \tag{1}$$

Think of a rotation of a "phasor" along a two dimensional circle centered at zero[1]. a and b ($\cos\varphi$ and $\sin\varphi$) then correspond to the x and y projections of the phasor, as shown in Figure 1. These are *real* and *imaginary* components of the complex exponential function

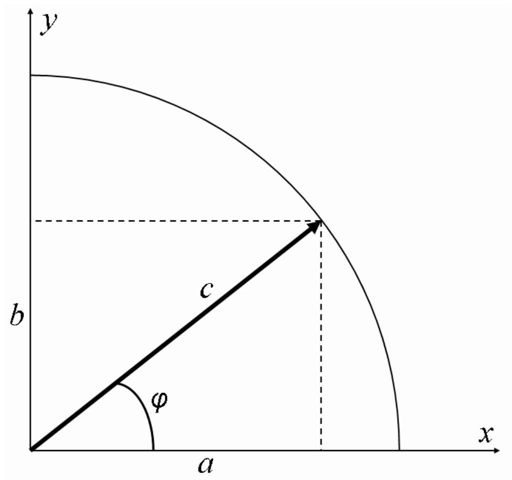

Figure 1: A point on a $x - y$ plane is defined by a complex number, c.

[1] Young & Freedman, "University Physics", Chapter 31.

(a unit vector of the complex plane), defined by the Euler relation[2]

$$e^{jx} = \cos x + j \sin x.$$

Respectively, $e^{-jx} = \cos x - j \sin x$ corresponds to a phasor having a negative imaginary (y) part.

For a stationary alternating current (ac) flowing through a circuit element, $i(t) = I_0 \cos \omega t$ [3], the motion of the current phasor is a continuous rotation with instantaneous angle ωt. The voltage across the element has generally a phase shift with respect to the current[4], $v(t) = V_0 \cos(\omega t + \phi)$. The same can be expressed in complex notations as

$$i(t) = \mathrm{Re}\left\{I_0 e^{j\omega t}\right\} = I_0 \cos \omega t \tag{2}$$

and

$$v(t) = \mathrm{Re}\left\{V_0 e^{j(\omega t+\phi)}\right\} = \mathrm{Re}\left\{V_0 e^{j\phi} e^{j\omega t}\right\} = \mathrm{Re}\left\{V e^{j\omega t}\right\}. \tag{3}$$

I_0 and V_0 above are real current and voltage amplitudes. $V = V_0 e^{j\phi}$ is *complex voltage*, which now has two parts - an amplitude and a phase (as defined by Eq. 1).

2 Ohm's law

Ohm's law for a resistor carrying a direct current (dc) is

$$V_0 = RI_0,$$

where the resistance is real and, therefore, the current and voltage are in phase. For a circuit carrying an ac, the Ohm's law must be modified to reflect the phase shift generally present between i and v (Eqs. 2,3). This is done by introducing a complex analogue of the resistance known as the impedance Z, such that the Ohm's law for ac becomes

$$V = ZI_0.$$

What is the form of Z for the common circuit elements R, L, and C?

[2]Wiki: "Euler's formula was proven for the first time by Roger Cotes in 1714 in the form $\ln(\cos(x) + i\sin(x)) = ix$. It was Euler who published the equation in its current form in 1748, basing his proof on the infinite series of both sides being equal. Neither of these men saw the geometrical interpretation of the formula: the view of complex numbers as points in the complex plane arose only some 50 years later."

[3]"Stationary alternating" here means that every next period is a repetition of the previous period. Almost always "ac" means a cos or sin form - well behavied functions under differentiation, in contrast to triangle or square waveforms.

[4]Following Y&F we choose the phase of the current to be the reference, i. e. $\phi_i = 0$.

3 Resistor

The voltage and current for a resistor of resistance R are in phase[5], and related for any instant in time by the Ohm's law

$$v(t) = Ri(t).$$

This means that

$$v = Ve^{j\omega t} = RIe^{j\omega t} \quad \text{or} \quad V = RI.$$

The resistive impedance is therefore

$$Z_R = R.$$

4 Inductor

For an ideal inductor ($R = 0$) the current-voltage relation is a consequence of the Faraday's law

$$v = L\frac{di}{dt}.$$

Using Eqs. 2 and 3, this yields a linear relation of Ohm-type[6] (again, with subscript "0" dropped for brevity)

$$V = j\omega LI.$$

The inductive impedance is then

$$Z_L = j\omega L$$

and has amplitude ωL and phase $\frac{\pi}{2}$, directed along $+y$ in Fig.1 for R along x. For an inductor with a non-vanishing resistance of the wire

$$v = Ri + L\frac{di}{dt} = i(R + j\omega L),$$

so the total impedance of the inductor is

$$Z_L = R + j\omega L.$$

This impedance is graphically shown in Fig. 2.

[5]Once again, subscript "0" refers to the fact that all phases are referenced to the phase of current. We will keep this in mind and drop the subscript for brevity of notations.

[6]Recall that $\frac{di}{dt}.e^{j\omega t} = j\omega e^{j\omega t}$.

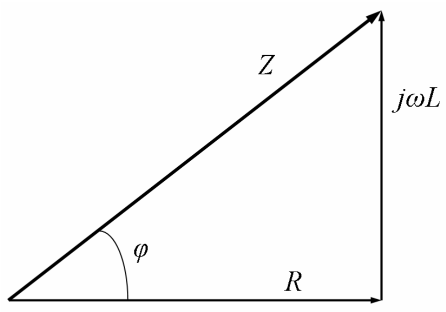

Figure 2: Inductive impedance as a vector sum of resistance and reactance.

5 Capacitor

The charge on an ideal capacitor is

$$q = Cv.$$

Differentiating both sides yields

$$\frac{dq}{dt} = i = C\frac{dv}{dt} = j\omega Cv = \frac{1}{Z_C}v,$$

and therefore $V = Z_C I$ with $(1 \equiv -j^2)$

$$Z_C = \frac{1}{j\omega C} = -\frac{j}{\omega C}. \tag{4}$$

The graphical interpretation of the capacitive impedance is straightforward: it is along the imaginary "j" axis (y axis in Fig. 1) and is opposite to the inductive impedance due to the minus sign in Eq. 4.

The total complex impedance of a series R-L-C circuit,

$$Z = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right),$$

is shown graphically in Fig. 3. The resistance R and reactance X correspond to the real and imaginary components of the impedance:

$$R = \mathrm{Re}\{Z\} \text{ and } X = \mathrm{Im}\{Z\}, \quad Z = R + jX,$$

where the reactance is

$$X = \omega L - \frac{1}{\omega C}.$$

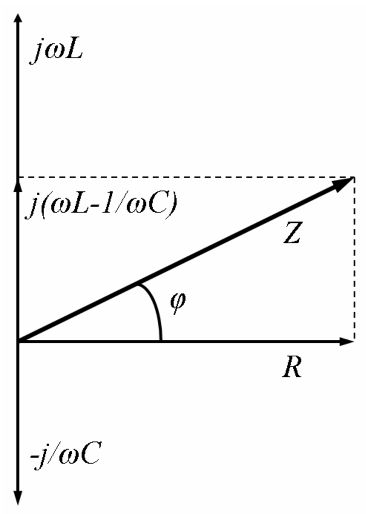

Figure 3: Impedance of a series R-L-C circuit with $|Z_C| = \frac{1}{2}|Z_L|$.

6 *ac* power

The instantaneous power delivered to a circuit element is[7]

$$p = vi = V\cos(\omega t + \phi)I\cos\omega t = VI\cos\phi\cos^2\omega t - VI\sin\phi\cos\omega t\sin\omega t.$$

It is convenient to characterize circuits by a time-independent average power. Averaging p, with $\cos^2\omega t = \frac{1}{2}$ and the second term vanishing, yields

$$P_{\text{av}} = \frac{1}{2}VI\cos\phi = V_{\text{rms}}I_{\text{rms}}\cos\phi.$$

In complex notations, the average power is expressed as follows[8]:

$$P_{\text{av}} = \text{Re}\left\{\frac{VI^*}{2}\right\} = \text{Re}\left\{\frac{Ve^{j(\omega t+\phi)}Ie^{-j\omega t}}{2}\right\} = \text{Re}\left\{\frac{Ve^{j\phi}I}{2}\right\} = \frac{VI}{2}\cos\phi. \tag{5}$$

The product in the braces ($P_{\text{av}} \to P$ for brevity)

$$S = \frac{1}{2}VI^* = V_{\text{rms}}I^*_{\text{rms}} = ZI_{\text{rms}}I^*_{\text{rms}} = ZI^2_{\text{rms}} = (R + jX)I^2_{\text{rms}} = P + jQ$$

is known the *complex power*, which has an *active* (P) and *reactive* (Q) parts. The modulus of the complex power is known as *apparent power*. To summarize, one distinguishes *complex*, *active*, *reactive*, or *apparent ac* power:

$$S = P + jQ = ZI^2_{\text{rms}},$$

[7] See Y&F §31.4 for more details.

[8] A^* denotes *complex conjugation*, which inverts the imaginary part of A.

$$P = \mathrm{Re}\{S\} = RI_{\mathrm{rms}}^2,$$
$$Q = \mathrm{Im}\{S\} = XI_{\mathrm{rms}}^2,$$
$$|S| = \sqrt{P^2 + Q^2} = |Z|I_{\mathrm{rms}}^2.$$

From Eq. 5 it is clear that the actual power dissipated in a circuit is associated with the *active power* (P) since the $\cos\phi$ (called the *power factor*) is zero for the reactive power component (Q).

Power matching in electric circuits means minimizing Q and, therefore, minimizing the reactance X. This is because for a given voltage, less current is needed to produce a given amount of power if the load is purely resistive. For a non-resistive component the current would flow in the reactive channel without producing heat (power).

7 Kirchoff's laws

The Kirchhoff's circuit laws continue to apply in the *ac* case, with the following generalizations: *dc* amplitudes are replaced with complex current amplitudes, $I_n \to I_n e^{j\phi_n}$, and *dc* voltages are replaced with complex voltages, $V \to Ve^{j\phi}$. Thus, for example, the voltage between points a and b of a circuit is related to the current by the element's complex impedance:

$$V_{ab} = Z_{ab} I_{ab}.$$

The *generalized junction rule* can be illustrated by a circuit with n elements of different resistance and reactance connected in parallel between points a and b (V_{ab} is fixed in this case). The total current is then a sum of the individual currents, which are phase shifted with respect to each other as determined by the impedances Zn:

$$i = \sum I_n \cos(\omega t + \phi_n).$$

Performing trigonometric summations is cumbersome. In complex notations the total current becomes a simple sum of complex numbers, $I = \sum I_n e^{j\phi_n}$. Indeed,

$$\mathrm{Re}\left\{Ie^{j\omega t}\right\} = \mathrm{Re}\left\{\sum I_n e^{j\phi_n} e^{j\omega t}\right\} = \sum \mathrm{Re}\left\{I_n e^{j(\omega t + \phi_n)}\right\} = \sum i_n(t) = i(t).$$

8 Filters

Filters are two-port circuits (signal in and out, Fig. 4) whose impedance is designed to select (filter) a certain frequency range out of a multi-frequency (broadband) signal. Filters can be low-pass, high-pass, or ban-pass (band-stop). These terms refer to the frequencies the filter passes through. Thus, a low-pass filter would transmit only low frequency signals from the input to the output.

The above three filter categories can be realized using RC, RL, and LC circuits. We use complex notations below to describe some common filter circuits.

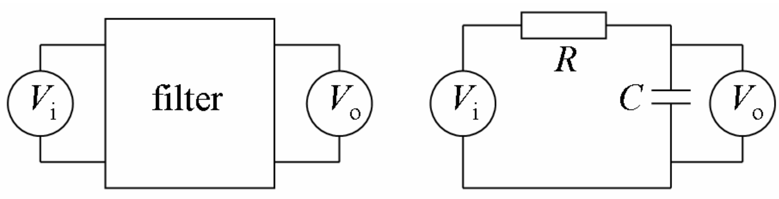

Figure 4: A general filter with two ports, input and output, and an RC filter.

8.1 R-C and C-R

A serial RC circuit is shown in Fig.4. V_i denotes a voltage source at the input and V_o a voltage meter at the output of the filter[9]. The series-RC impedance is

$$Z = R - \frac{j}{\omega C}.$$

The output voltage - the voltage across the capacitor - is given by the current in the circuit multiplied by the impedance of the capacitor:

$$V_o = I Z_C = \frac{V_i}{Z} Z_C = \frac{V_i}{R - j/(\omega C)} \left(-\frac{j}{\omega C} \right) = \frac{j}{j - \omega RC} V_i.$$

The transmission, or filter coefficient then becomes

$$T = \frac{V_o}{V_i} = \frac{j}{j - \omega RC},$$

and its amplitude

$$|T_{RC}| = \left| \frac{V_o}{V_i} \right| = \sqrt{\frac{j}{j - \omega RC} \left(\frac{j}{j - \omega RC} \right)^*} = \frac{1}{\sqrt{1 + (\omega RC)^2}}. \tag{6}$$

For low frequencies, $\omega \to 0$, the signal is fully transmitted, $|T| \to 1$. At high frequencies, $\omega \to \infty$, the signal is completely suppressed, $|T| \propto \omega^{-1} \to 0$. This is the action of a *low-pass* filter, used in electronics to filter out high frequencies.

The cutoff frequency of the filter is taken to be at the point where one half of the power is transmitted,

$$P_{\text{cutoff}}(\omega_c) = \frac{1}{2} P_{\max},$$

with $P_{\max}$ in this case corresponding to $P(\omega \to 0)$. Since $P \propto V^2$, the cutoff frequency is where $|T| = 1/\sqrt{2}$. From Eq. 6, this condition corresponds to $\omega_c RC = \omega_c \tau_{RC} = 1$, or

$$f_c = \frac{1}{2\pi RC}.$$

[9]The input impedance of the voltmeter is assumed to be infinite - a excellent approximation in most cases.

The cutoff frequency defined this way is also known as *the -3 dB point*[10]

Exchanging the R and C in the series RC circuit, with V_o now measured across the resistor, the circuit's frequency response becomes

$$|T_{CR}| = \left|\frac{R}{R - j/(\omega C)}\right| = \sqrt{\frac{\omega RC}{\omega RC - j}\left(\frac{\omega RC}{\omega RC - j}\right)^*} = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}.$$

For low frequencies, $\omega \to 0$, the signal is suppressed, $|T| \propto \omega \to 0$. At high frequencies, $\omega \to \infty$, the signal is fully transmitted, $|T| \to 1$. This is the action of a *high-pass* filter, used in electronics to filter out *dc*.

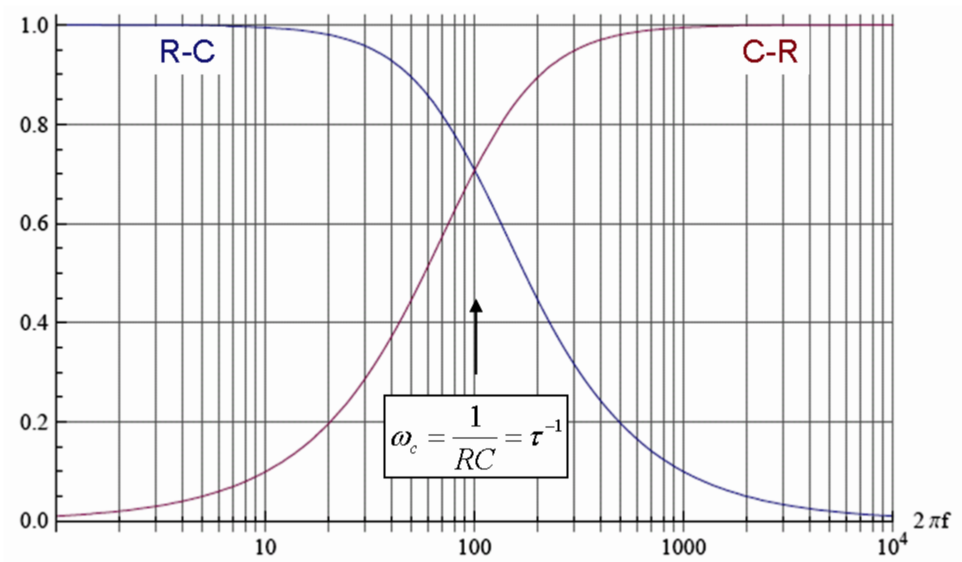

Figure 5: $T(\omega)$: low- and high-pass filtering using R-C and C-R circuits.

8.2 R-L and L-R

If the capacitor in Fig.4 (right) is replaced with an inductor, then the filter factor becomes

$$|T_{RL}| = \sqrt{\frac{j\omega L}{R + j\omega L}\left(\frac{j\omega L}{R + j\omega L}\right)^*} = \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}},$$

which represents a high-pass filter, similar to the C-R circuit described above (red curve in Fig.5). The difference is that the cutoff condition $|T| = 1/\sqrt{2}$ now corresponds to $\omega_c = R/L$, so time constant

$$\tau_{RL} = \frac{L}{R}.$$

[10] A voltage ratio in *decibel* is defined as $X_{dB} = 20\log_{10}\frac{X}{X_0}$. In our case of the cutoff frequency, $20\log_{10}\frac{1}{\sqrt{2}} = -3.01$ dB.

Exchanging the positions of the R and L in the circuit results obviously in a low-pass filter of the R-C type, with

$$|T_{LR}| = \sqrt{\frac{R}{R+j\omega L}\left(\frac{R}{R+j\omega L}\right)^*} = \frac{1}{\sqrt{1+(\omega L/R)^2}}.$$

8.3 LC in series

In the RC and RL circuits above only one element (C or L) had a frequency dependent impedance (reactance, X_C or X_L). Therefore the circuit impedance either increased or decreased with frequency. In LC circuits, on the other hand, one expects a competition between a rising reactance of the inductor and diminishing reactance of the capacitor, as the frequency is increased. This should result in a non-monotonic behavior of the filter, i.e. maxima or minima in current or voltage.

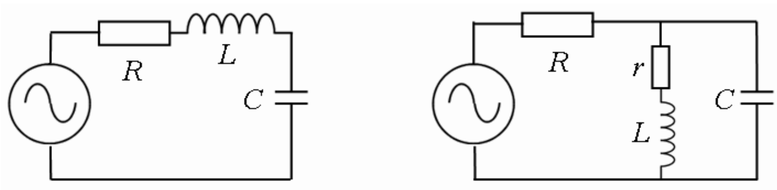

Figure 6: Series and parallel LC resonant circuits.

The left panel of Fig.6 shows a series RLC circuit. The current is given by the voltage supplied by the source divided by the total impedance of the circuit,

$$I = \frac{V}{Z} = \frac{V}{R+j[\omega L - 1/(\omega C)]}. \tag{7}$$

The current is maximum, $I_0 = V/R$, when $\omega_0 L = 1/(\omega_0 C)$, from which the *LC resonance* frequency is

$$\omega_0 = (LC)^{-\frac{1}{2}}.$$

The *quality factor* is defined as

$$Q \equiv \frac{\sqrt{L/C}}{R} = \omega_0 \frac{L}{R}.$$

Using this definition, the current of Eq. 7 can be rewritten as

$$I = I_0 \left[1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]^{-1},$$

and its amplitude

$$\left|\frac{I}{I_0}\right| = \left[1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2\right]^{-\frac{1}{2}}. \tag{8}$$

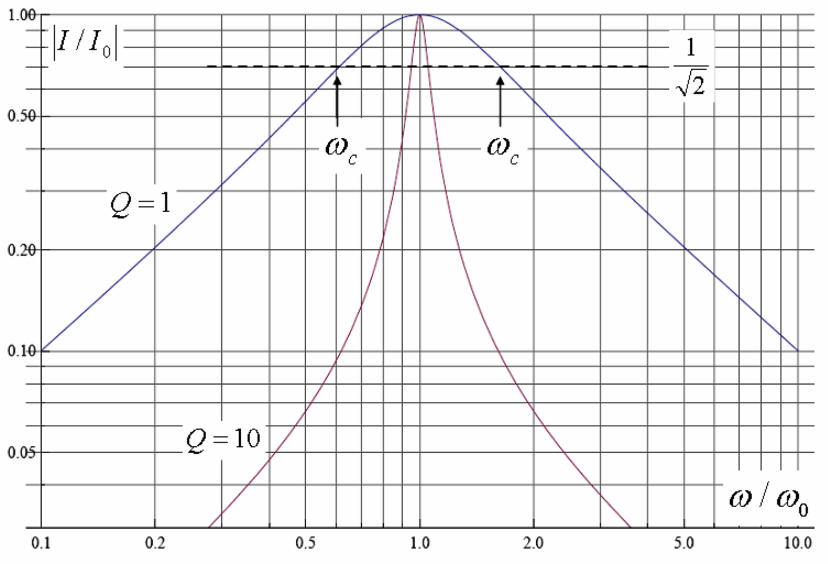

Figure 7: Normalized current of Eq.8 for a series LC circuit.

This function is plotted in Fig. 7 for $Q = 1, 10$. The cutoff frequencies defined by $T = 1/\sqrt{2}$ correspond to

$$Q\left(\frac{\omega_c}{\omega_0} - \frac{\omega_0}{\omega_c}\right) = \pm 1,$$

with the left and right cutoff

$$\frac{\omega_{c\mp}}{\omega_0} = \sqrt{\left(\frac{1}{2Q}\right)^2 + 1} \mp \frac{1}{2Q},$$

and the relative resonance width ($\Delta\omega = \omega_{c+} - \omega_{c-}$)

$$\frac{\Delta\omega}{\omega} = \frac{1}{Q}.$$

The voltages across the individual circuit elements are obtained by multiplying the total current (Eq. 7) by the respective impedance. For example, for the capacitor

$$V_C = IZ_C = \frac{I}{j\omega C}.$$

Clearly, the sharply peaked current versus frequency (for high Q) results is a high voltage across the capacitor only for a narrow band in frequency (ωC is a monotonous function). This is the principle behind band-pass filtering in great many electronic circuits and systems we use today.

8.4 LC in parallel

The analysis of the parallel LC circuit is very similar to the one we have just performed, and is left as a home task. Consider the circuit layout shown in the right panel of Fig.6. This layout is identical with your filter layout in the LabVIEW lab.

Your model of the impedance and the quality factor should be adjusted to fit your experimental data. Proceed by recognizing that the parallel connection of the L and C branches is connected in series with R. Furthermore, the resistance of the inductor wire (r) cannot be neglected, and forms a series connection with the inductive reactance. Your output voltage is measured across C.